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Large Deviations and Multifractal Analysis for Expanding Countably-Branched Markov Maps

Large Deviations and Multifractal Analysis for Expanding Countably-Branched Markov Maps

By

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Abstract

We will use the theory of thermodynamic formalism for countable Markov shifts to pose and solve problems in multifractal analysis and large deviations. We start with an introduction outlining results in thermodynamic formalism, multifractal analysis, and large deviations in Chapter 1. We state necessary concepts and results from dynamical systems, ergodic theory, thermodynamic formalism, dimension theory, and large deviations in Chapter 2.

In Chapter 3, we consider the multifractal analysis for Gibbs measures for expanding, countably branched Markov maps. We will find conditions for the multifractal spectrum to have various numbers of phase transitions. Finally, in Chapter 4, we consider an expanding, countably-branched Markov map T_λ, the countable Markov shift Σ_A, and a locally Hölder potential $f : \Sigma_A \to \mathbb{R}$. The behaviour of the dynamical system $(T_\lambda, (0,1])$ depends on the value of λ. We will aim to form a large deviation principle for $\frac{S_n f}{n}$ for a fixed $\lambda \in (\frac{1}{2}, 1)$ and we will discuss the method for forming a such a principle for $\lambda \in \left(0, \frac{1}{2}\right]$ in Chapter 4's introduction.

Dedication and acknowledgements

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Author's declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Chapter 1

Introduction

1.1 Dynamical Systems and Ergodic Theory

This thesis will consider problems in multifractal analysis and large deviations from the point of view of dynamical systems and ergodic theory. We discuss the historical and contemporary developments in dynamical systems and ergodic theory. Our discussion is by no means exhaustive. Recently, there has been a proliferation of results in other areas of mathematics, which use ergodic theory, such as combinatorics, number theory, probability, and fractal geometry. Of course, many results have also developed within areas of dynamics, such as non-uniformly hyperbolic dynamical systems and Teichmüller theory. Now, we will discuss a development that sparked more than a century of theory in dynamical systems.

1.1.1 The Beginnings of These Areas

In its simplest form, a dynamical system is made of a set and its dynamics (often, a map on this set). Here is a famous quote by Poincaré: "It is by logic we prove, it is by intuition that we invent." The developments in dynamical systems over the past 128 years and ergodic theory around the past 80 years have been the product of logic and intuition. Henri Poincaré won King Oscar II of Norway and Sweden's prize for his work on the three body problem (about the possible orbits of three planets attracting each other). This led to his prominence as a French mathematician and the proof of his famous recurrence theorem (see Theorem 2.2.6). Given a space X, a finite, invariant measure m, and a set $E \subset X$ such that $m(E) > 0$, his recurrence theorem states that the orbit of an $m-$typical point $x \in E$ will eventually return to E. Poincaré's theorem can be used to construct spaces from recurrent points to analyse properties of a dynamical system. This is discussed in the introduction to Chapter 4.

Poincaré's work leads to developments in ergodic theory such as the Birkhoff Ergodic Theorem. The word "ergodic" comes from the Greek words "ergon," meaning work, and "odos," meaning path or way. This connects to Boltzmann's hypothesis, in statistical mechanics, which states that the time and space average of a system are equal. This hypothesis was formalised and became the Birkhoff Ergodic Theorem. Of course, a dynamical system needs a finite measure and a measure preserving transformation before the Birkhoff Ergodic Theorem is applied. The necessity of a finite measure is proven later by Aaronson.

Using Birkhoff's theory, Sinai analysed dynamical systems with measures. The long term behaviour of these systems is hard to predict because orbits will wander. His work with Kolmogorov for his Masters (the PhD in the USSR at that time) lead to their development of measure-theoretic entropy. This function quantifies the amount of "chaos" or disorder in a system. For a discussion and formal definition of chaos, see Devaney [D$^+$89]. Later, Bowen [BC75] defined the concept of topological entropy, which gives another value to quantify chaos. His work connected dynamical systems and ergodic theory to statistical mechanics. For more discussion on the history of dynamical systems and ergodic theory, see Badino [Bad06], O'Connor and Robertson [OR01], [OR15], and [OR14], and Gray, Hansen, and Holmes [GHH10]. An area of ergodic theory, stemming from Birkhoff and Bowen's research, is thermodynamic formalism.

1.2 Thermodynamic Formalism

We will use thermodynamic formalism for problems in dimension theory and large deviations. Entropy and pressure are connected to the decay rate of Birkhoff sums and are used in multifractal analysis and large deviations. Both areas are discussed in further detail on the next two sections. In multifractal analysis, analysis of the pressure function yields a Legendre transform, which is the expression for the standard multifractal spectrum. In large deviations, using the pressure function helps one to form a rate function.

Bowen and Ruelle developed the thermodynamic formalism for finite state Markov shifts. Ruelle [Rue04] analysed the thermodynamic formalism for equilibrium statistical mechanics and in particular, spin lattice systems. Roughly speaking, in this setting, one has a state space made of a lattice and a configuration space, which is the shift space. Thermodynamic formalism, in the context of statistical mechanics, considers problems such as the analysis of the state of water changing from solid to liquid. This phenomenon is called a phase transition.

Bowen [BC75] extended his results on thermodynamic formalism to the context of Anosov and Axiom A diffeomorphisms. Expanding, finitely-branched Markov maps on a compact spaces are examples of one dimensional Axiom A diffeomorphisms. To understand the structure of the orbits of finitely-branched expanding maps, we use thermodynamic formalism. Gibbs states come from statistical mechanics. We can understand the limiting behaviour of the orbit of a typical point by using a Gibbs measure. He finds conditions for the existence of a Gibbs measure and defines the notions of pressure (or negative free energy in the context of statistical mechanics) and equilibrium state.

To work on problems on countably branched expanding Markov maps and countable Markov shifts, we must use Sarig's thermodynamic formalism [Sar99] for these shifts. We cannot apply the thermodynamic formalism for finite state Markov shifts, developed by Bowen and Ruelle, to the setting of countable state Markov shifts because these spaces are not compact and lack the combinatorial properties of finite state Markov shifts. For instance, Gibbs and equilibrium states are not guaranteed to exist for potentials on countable state Markov shifts. However, pressure of potentials on these shifts can be approximated by using compact subshifts. Problems in large deviations and multifractal analysis have been posed in the setting of finitely-branched expanding Markov maps and their finite state Markov shifts. This thesis concentrates on these problems in the countable case.

Sarig developed the thermodynamic formalism for countable Markov shifts [Sar99]. Note that there were ergodic theorists who studied this type of thermodynamic formalism for particular expanding, Markov maps and their countable shifts. His work [Sar03] established criteria for the existence of Gibbs and equilibrium states for potentials on these shifts. The existence of Gibbs measures for potentials on countable shifts is crucial for Chapter 3 because it allows us to prove that the multifractal spectrum might have non-analytic points or phase transitions. The pressure function on such a shift might have non-analytic points. In summary, Sarig's theory enables ergodic theorists to analyse problems on countably branched expanding Markov maps. For a thorough discussion of Sarig's work, please see his survey [Sar15].

We proceed by discussing two applications of thermodynamic formalism: multifractal analysis and large deviations. We will later analyse the similarities and differences of both areas in Subsection 1.4.3. First, we will discuss a form of multifractal analysis. In Chapter 3, we will consider a Gibbs measure on a countable Markov shift. Sarig's works help us analyse the phase transitions of the standard multifractal spectrum for this Gibbs measure.

1.3 Multifractal Analysis

We discuss research done in multifractal analysis. A standard form of multifractal analysis involves analysing the scaling behaviour of a measure. We will work on a problem in this form of multifractal analysis in Chapter 3. Take a Gibbs measure on a (finite or countable) shift space Σ_A. Consider the set X_α of $x \in \Sigma_A$ that have local dimension α. The multifractal spectrum is the function that sends each α to the Hausdorff dimension of X_α. We can also define the multifractal spectrum in other settings such as $\mathbb{R}$ rather than Σ_A. This function quantifies the concentration of a measure on these sets. Our discussion on multifractal analysis is divided into the settings of finite and countable state Markov shifts. We will briefly discuss multifractal analysis in other settings.

1.3.1 Multifractal Analysis for Measures on Finite State Markov shifts

Now, we discuss past work on multifractal analysis in the setting of finite state Markov shifts. Rand [Ran89] considers a cookie-cutter, which is a uniformly hyperbolic map. He uses thermodynamic formalism on a finite state Markov shift to prove that the multifractal spectrum is analytic everywhere. Next, we discuss the work of Cawley and Mauldin [CM92]. They consider a fractal constructed by taking an iterated function system on a self-similar set (which yields a self-similar measure). Falconer [Fal04] gives more details on the construction of such a fractal. Cawley and Mauldin model an iterated function system with a finite state Markov shift. Their methodology to prove that the multifractal spectrum is analytic everywhere involves geometric arguments. Pesin and Weiss [PW97] consider a uniformly expanding map and form an expression for the multifractal spectrum. They use a combination of thermodynamic formalism and a covering argument to prove that the multifractal spectrum is analytic everywhere.

1.3.2 Multifractal Analysis for Measures on Countable State Markov shifts

The multifractal spectrum for measures on a countable full shift is not always analytic. Iommi [Iom05] forms a general formula for the multifractal spectrum with respect to a Gibbs measure on the countable shift and gives conditions for its analyticity. Hanus, Mauldin, and Urbański also consider the multifractal spectrum in the setting of a countable, conformal iterated function system modelled by a countable Markov shift. Their paper [HMU02] gives additional conditions to prove that the multifractal spectrum is analytic.

We remark that Iommi and Jordan's work [IJ15a] uses a similar setting compared to Chapter 3. They analyse the phase transitions of the pressure function and the multifractal spectrum. In [IJ15a], Iommi and Jordan assume that two potentials $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are bounded and $\lim_{x\to 0} \frac{\psi(x)}{\log|T'(x)|} = \infty$ (in our paper, the limit equals 1). In their case, the multifractal spectrum has 0 to 2 phase transitions. In their other work [IJ13], they take g to be a continuous function defined on the range of the suspension flow. Then, they prove that the map $t \mapsto \mathscr{P}(tg)$ has 0 to 1 phase transition when the roof function dominates the floor function. Using this result, Iommi and Jordan prove that the multifractal spectrum has 0 to 2 phase transitions in [IJ13].

Their paper [IJ15b] considers expanding, countably-branched Markov maps on $[0,1]$ and level sets generated by Birkhoff averages. This paper uses results from [IJ13] and [IJ15a]. Their results include a variational characterisation of the multifractal spectrum and the existence of 0 to 2 phase transitions for $f_\mu(\alpha)$ when α_{lim}, a ratio involving the potentials ϕ and ψ, equals 0. In contrast, Chapter 3 assumes that $0 < \alpha_{\text{lim}} \leq \infty$ or does not exist. Their paper [IJ15b] also proves results on the multifractal analysis of suspension flows. Let T be an expanding interval map and g be a continuous function defined on the range of the suspension flow. Iommi and Jordan [IJ15b] prove that the Birkhoff spectrum with respect to g has two phase transitions if the roof function dominates the geometric potential $\log|T'|$.

1.3.3 The Multifractal Spectrum in Other Settings

We briefly discuss examples of phase transitions for the multifractal spectrum in other settings. Researchers have studied phase transitions for non-uniformly expanding interval maps that have neutral fixed points. They use thermodynamic formalism and respectively form formulae for the multifractal spectrum. Olivier's paper [Oli00] considers a cookie cutter on $[0,1]$ and takes an induced map defined on a Cantor set generated by this cookie cutter. Nakaishi's paper [Nak00] considers piecewise interval maps on $[0,1]$, such as the Farey map, and an induced transformation generated by these maps. We note that Nakaishi's paper is related to a paper by Pollicott and Weiss [PW99]. Pollicott and Weiss's paper considers a class of non-uniformly hyperbolic maps called EMR maps (see Page 149 of [PW99]). Examples of such maps include the Pomeau-Maneville map and the continued fraction map. They use an inducing scheme to get an equilibrium state for their multifractal analysis. Finally, Pollicott and Weiss analyse the multifractal spectra resulting from these maps and conclude that the domain of the multifractal spectrum can be bounded. We will now discuss another application of thermodynamic formalism: large deviations.

1.4 Large Deviations

Large deviations originated in probability. In that context, one analyses the frequency of averages of n i.i.d. random variables that are away from the expectation. Cramer's Theorem states that there exists a Legendre transform that measures the exponential decay rate of this frequency. Yuri Kifer [Kif90] and Lai-Sang Young [You90] were among the first to analyse this problem dynamically. This could be done because hyperbolic dynamical systems have similar behaviour compared to i.i.d. random variables. We will later define the dynamical definition of a rate function, given by Definition 2.8.5. Our discussion of research in large deviations is divided into work on uniformly and non-uniformly hyperbolic dynamical systems.

1.4.1 Large Deviations in Uniformly Hyperbolic Dynamics

We start with the classical setting of large deviations of uniformly hyperbolic dynamics. Young [You90] considers a continuous map $f: X \to X$ on a compact metric space X. She considers two subsets of the space of continuous functions $C(X,\mathbb{R})$, some of which satisfy the specification property, and a reference, Borel measure m on X. Take the potential $\phi = \log|f'|$. She forms a large deviation principle for $\frac{1}{n}S_n\phi$. Her rate function, a conditional variational principle, uses a function $\xi \in C(X,\mathbb{R})$ that is closely related to m. Furthermore, Young finds large deviation estimates on Axiom A attractors and finite state Markov shift spaces as well as rates of escape in invariant sets. In contrast, Kifer [Kif90] takes a family of probability spaces $(\Omega_\lambda, F_\lambda, P_\lambda)$, each with a measurable map ζ^λ. Then, he uses probabilistic methods to find large deviation estimates for ζ^{λ} with respect to the reference measure P_{λ} on $\mathbb{R}^n$. Then, he applies his large deviation results to discrete expanding maps, flows, and stochastic processes.

This thesis uses the dynamical approach to large deviations. Keller and Nowicki [KN92] take an expanding Collet-Eckmann map $T:[0,1]\to[0,1]$ and the function $F=-\log|T^{'}|$. Let Per_n be the set of $n-$periodic points with respect to T. Keller and Nowicki's two large deviation estimates for $S_N F$ respectfully use Lebesgue measure and a measure ν_n, such that $\nu_n(Per_n)=1$, as the reference measure. They use thermodynamic formalism and Markov extension.

1.4.2 Large Deviations in Non-Uniformly Hyperbolic Dynamics

Large deviation principles can also be formed for non-uniformly hyperbolic dynamical systems. Keller and Nowicki [KN92], Pollicott and Sharp [PS09], and Chung and Takahasi analyse maps that are similar to T_{λ} (see (4.1.1)) in Chapter 4. Before analysing Pollicott and Sharp's results, we briefly define the terms level 1 and level 2 large deviation principle. Our definition of a large deviation principle (see Definition 2.8.5) is the definition for a level 1 large deviation principle. Informally, a level 1 large deviation principle finds the decay rate of sets whose points have Birkhoff averages away from the expected mean. Although we will not formally define the term: level 2 large deviation principle, we note that a level 2 large deviation principle considers an average of Dirac measures and finds the decay rate of sets in the weak star topology for measures. According to Pollicott and Sharp, level 1 large deviation principles 'are for functions' and level 2 large deviation principles 'are for measures.' See Pollicott and Sharp's introduction [PS09] for a discussion on level 1 and level 2 large deviation principles.

Pollicott and Sharp [PS09] consider a generalised version $T:I\to I$ of the Pomeau-Manneville map, its acip μ, and a Hölder potential $f:I\to\mathbb{R}$. Then, they form a large deviation principle for $\frac{S_n f}{n}$ with respect to μ by using the transfer operator method and Dirac measures. As stated in their paper [PS09], they find level 2 large deviation results to form their level 1 large deviation principle.

A more recent paper by Chung and Takahasi [CT17] use techniques similar to Chapter 4 to form their large deviation principles. Chung and Takahasi [CT17] consider an S-unimodal map $f:[0,1]\to[0,1]$ that has an acim. They continuous potential $\phi:[0,1]\to\mathbb{R}$. Then, Chung and Takahasi form a large deviation principle for $\frac{S_n\phi}{n}$. They use a “specification-like property" in the argument for the upper bound for their large deviation principle. They “glue the orbits of a tail set and a nice interval I," which is similar to the argument for the lower bound of our rate function (see Section 4.8). However, they apply this to their inducing scheme for their map f. In contrast, the Poincaré recurrence theorem cannot be applied to our map T_{λ}, when $\lambda\in(\frac{1}{2},1)$ (in Chapter 4) due to the behaviour of Lebesgue typical points (see Theorem 4.1.4). Hence, we would are not able to use an inducing scheme in Chapter 4.

Chung and Takahasi also find properties of their rate function in Theorems B and C (Page 6). Furthermore, they consider a set of Lipschitz functions. Distinctively, they form a uniform upper bound (Proposition 3.2, Page 12) as part of a large deviation principle for this set. They aimed to find the largest set of functions that have large deviation principles.

Luc-Rey Bellet and Young [RBY08], Melbourne and Nicol [MN08], Varandas [Var12], Climenhaga, Thompson, and Yamamoto [CTY17], and many more form large deviation principles for non-uniformly hyperbolic dynamical systems. See their works for more details. Because we have discussed past research in multifractal analysis and large deviations, we will discuss their similarities and differences.

1.4.3 Relating Multifractal Analysis to Large Deviations

Take a countable, expanding Markov map $T:[0,1]\to[0,1]$, its associated countable full shift $\Sigma := \mathbb{N}^{\mathbb{N}}$, and the coding map $\pi:\Sigma\to[0,1]$. Consider two locally Hölder potentials $\phi:\Sigma\to\mathbb{R}^-$ and $\psi:\Sigma\to\mathbb{R}^+$ such that $\psi := \log|T'\circ\pi|$. In Chapter 3, we will take the Gibbs measure μ for ϕ and a metric that is closely related to ψ. Denote $|S|$ as the diameter of a set $S\subset\Sigma$. We will find that

$$\left\{x\in\Sigma : \lim_{k\to\infty}\frac{\mu([x_1,\dots,x_{k-1}])}{|[x_1,\dots,x_{k-1}]|}=\alpha\right\}=\left\{x\in\Sigma : \lim_{k\to\infty}\frac{S_k\phi(x)}{-S_k\psi(x)}=\alpha\right\}. \tag{1.4.1}$$

Take the set

$$X^s_\alpha := \left\{x\in\Sigma : \lim_{k\to\infty}\frac{S_k\phi(x)}{-S_k\psi(x)}=\alpha\right\}.$$

Finally, we will count the number of possible phase transitions for the Birkhoff spectrum $\alpha\mapsto\dim_H(X^s_\alpha)$ under certain conditions.

Because of Equation (1.4.1), $\alpha\mapsto\dim_H(X^s_\alpha)$ is also the standard multifractal spectrum. We note that Iommi, Jordan, and Todd find similar results. They use the map T_λ (see Equation (4.1.1)) to analyse the Lyapunov spectrum (i.e. the Birkhoff spectrum when $\psi := \log|T_\lambda'|$) for a Hölder potential ϕ and that choice of ψ. Proposition 4.4 and Figure 1 of their paper [IJT17] prove that the Birkhoff spectrum has a phase transition.

Using a different perspective, we could have also formed a large deviation principle for $\frac{S_k\phi}{-S_k\psi}$. However, we note that Legendre transforms and the pressure function are often used to form expressions for the standard multifractal spectrum and rate functions in large deviation principles. Fix an $\alpha>\frac{\int\phi\,\mathrm{d}\mu}{\int-\psi\,\mathrm{d}\mu}$. For each $k\in\mathbb{N}$, take the set

$$X^s_{\alpha,k} := \left\{x\in\Sigma : \frac{S_k\phi(x)}{-S_k\psi(x)}\geq\alpha\right\}.$$

Then, we would find an expression for $\lim_{k\to\infty}\frac{1}{k}\log\mu\left(X^s_{\alpha,k}\right)$ for each of these α. In this case, the function R, such that

$$R(\alpha)=\sup_{\nu\in M_\sigma(\Sigma)}\left\{\int-\psi\,\mathrm{d}\nu+h(\nu) : \frac{\int\phi\,\mathrm{d}\nu}{\int-\psi\,\mathrm{d}\nu}\geq\alpha\right\}=\lim_{k\to\infty}\frac{1}{k}\log\mu\left(X^s_{\alpha,k}\right) \tag{1.4.2}$$

for each $\alpha>\frac{\int\phi\,\mathrm{d}\mu}{\int-\psi\,\mathrm{d}\mu}$, has no phase transitions. The setting for Equation (1.4.2) contrasts with the setting in Chapter 4 because an invariant Gibbs measure does not exist in that setting.

We note that ergodic theorists find phase transitions for various functions by using thermodynamic formalism. We note that Iommi and Todd analyse non-Markov maps, similar to T_λ (see Equation 4.1.1 in Chapter 4), and find results on the phase transitions of pressure functions.

1.5 Summary of Results

Finally, we give an overview of this thesis and its main aims. Chapter 2 states background information on dynamical systems, ergodic theory, thermodynamic formalism, multifractal analysis, dimension theory, iterated function systems, and large deviations.

Consider a Gibbs measure on a finite state full shift. The multifractal spectrum with respect to this measure is analytic. However, the countable case is quite different. This leads to the problem analysed in Chapter 3, which is described as follows. First, take a countable Markov shift Σ, which is topologically mixing and satisfies the BIP property. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to $-\psi$. The locally Hölder potential $\phi : \Sigma \to \mathbb{R}^-$ has a Gibbs state μ. Using ψ, we will choose a general metric that satisfies Inequality (3.1.1). This choice of metric allows us to use the following definition for the multifractal spectrum, which will be proven to be equivalent to the standard definition for the multifractal spectrum (see Proposition 3.5.4 and Definitions 3.5.6 and 3.5.3). Iommi [Iom05] develops the following equivalent definition (in our setting) for the multifractal spectrum.

Take the values

$$\alpha_{\inf} = \inf\left\{ \lim_{m\to\infty} \frac{\log \mu([x_1, x_2, \ldots, x_m])}{\log |[x_1, x_2, \ldots, x_m]|} : [x_1, x_2, \ldots, x_m] \subset \Sigma \right\} \text{ and}$$

$$\alpha_{\sup} = \sup\left\{ \lim_{m\to\infty} \frac{\log \mu([x_1, x_2, \ldots, x_m])}{\log |[x_1, x_2, \ldots, x_m]|} : [x_1, x_2, \ldots, x_m] \subset \Sigma \right\}.$$

Consider

$$X_\alpha^s := \left\{ x \in \Sigma : \lim_{m\to\infty} \frac{\log \mu([x_1, x_2, \ldots, x_m])}{\log |[x_1, x_2, \ldots, x_m]|} = \alpha \right\}$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$. Lemma 3.5.5 proves that $X_\alpha^s = \emptyset$ if $\alpha \notin [\alpha_{\inf}, \alpha_{\sup}]$. Because the symbolic and local dimension are equal on every set with the exception of a set of small Hausdorff dimension (see Proposition 3.5.4), the multifractal spectrum f_μ (see Proposition 3.5.7) is given by

$$f_\mu(\alpha) = \dim_H(X_\alpha^s)$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$. We will use Iommi's expression for the multifractal spectrum (see Proposition 3.5.7 which is Theorem 4.1 in [Iom05]).

Consider an arbitrary $\bar{N} := (N, x_2, x_3, \ldots) \in \Sigma$ for each $N \in \mathbb{N}$. We will consider the value $\alpha_{\lim} := \lim_{N\to\infty} \frac{\phi(\bar{N})}{-\psi(\bar{N})}$. See Subsection 3.3.1 (and in particular, see Inequalities (3.3.2), (3.3.3), and (3.3.6) and Equations (3.3.4) and (3.3.5)) for a discussion on the motivation of its use and how it is related to our potentials' conditions. See Chapter 3 for the proofs of Theorems 3.1.5 and

3.1.6, which are respectively stated as follows. We will use Sarig's thermodynamic formalism (see Sarig [Sar99]) and Iommi's results [Iom05] on the multifractal spectrum to prove our results. We will prove that the multifractal spectrum has 0 to 3 phase transitions (non-analytic points) if $\alpha_{\lim} \in (0,\infty)$.

Theorem 1.5.1. *Assume that (Σ,σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to $-\psi$ and the potentials were chosen so that $0 < \alpha_{\lim} < \infty$. Denote μ as the Gibbs state for ϕ.*

1. *There exist intervals A_i such that $f_\mu(\alpha)$ is analytic on each of their interiors.*
2. *The interval $(\alpha_{\inf}, \alpha_{\sup}) = \cup_{i=1}^{j} A_i$ such that $j \in \{1,2,3,4\}$.*
3. *The multifractal spectrum is concave on $(\alpha_{\inf}, \alpha_{\sup})$, has a maximum at a single point, and has zero to three phase transitions.*

We will also prove that the multifractal spectrum has 0 to 1 phase transition if $\alpha_{\lim} = \infty$.

Theorem 1.5.2. *Assume that (Σ,σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to $-\psi$ and the potentials were chosen so that $\alpha_{\lim} = \infty$. Denote μ as the Gibbs state for ϕ.*

1. *There exist intervals A_i such that $f_\mu(\alpha)$ is analytic on each of their interiors.*
2. *The interval $(\alpha_{\inf}, \alpha_{\sup}) = \cup_{i=1}^{j} A_i$ such that $j \in \{1,2\}$.*
3. *The multifractal spectrum is concave on $(\alpha_{\inf}, \alpha_{\sup})$ and has zero to one phase transition.*

We will use thermodynamic formalism on countable state Markov shifts to find the number of possible phase transitions for the multifractal spectrum. Then, we apply these results to the Gauss map G, which is modelled by $\mathbb{N}^{\mathbb{N}}$. We consider the potential $\psi := \log|G' \circ \pi|$ with the coding map $\pi : \Sigma \to [0,1]$ given by the continued fraction map. The coding map allows us to apply our results about the multifractal spectrum's phase transitions, proven on (Σ,σ), to $([0,1],G)$. In particular, there is a conjugacy (up to countably many points) between G and σ.

Consider Young's [You90] result (see Theorem 2.8.3) on large deviations for an expanding, finitely-branched map. Young obtains conditional variational principles for her both of her bounds. In contrast, the behaviour of our map T_λ in Chapter 4 (see Equation 4.1.1) leads to a weighted conditional variational principle. In Chapter 4, we consider an expanding, countably-branched Markov map $T_\lambda : (0,1] \to (0,1]$ (see Equation 4.1.1) for a fixed $\lambda \in (\frac{1}{2},1)$. This map has a Markov partition $\{R_1,...\}$ such that $R_n := (\lambda^n, \lambda^{n-1}]$ for each $n \in \mathbb{N}$. Consider its associated shift space Σ_A (see Equation 4.1.3) and coding map $\pi : \Sigma_A \to (0,1)$ such that $\pi^{-1}(R_n) := [n]$. In particular, there is a conjugacy (up to countably many points) between T_λ and σ. Stratmann and Vogt (see [SV97]) first analysed the dimension theoretic properties associated to this map. Bruin and Todd [BT12] analyse the thermodynamic formalism of this map.

We will now state the full setting for our problem. Denote l for Lebesgue measure. We will take $m = l \circ \pi$ as the reference measure. Consider $\bar{N} := (N, N, N, \ldots)$ for each $N \in \mathbb{N}$. Take the locally Hölder potentials $\phi_\lambda := -\log|T_\lambda \circ \pi|$ and $f : \Sigma_A \to \mathbb{R}$ such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty, \infty)$. Our aim will be to form a large deviation principle for $\frac{S_n f}{n}$.

We will take

$$L := \lim_{N\to\infty} f(\bar{N}) \text{ and } \alpha_{\text{sup}} := \max_{\nu \in M_\sigma(\Sigma_A)} \left\{ \int f \, \mathrm{d}\nu \right\}.$$

We will prove that $m-$typical sequences are transient and

$$\lim_{n\to\infty} \frac{S_n f(x)}{n} = L$$

for each $m-$typical $x \in \Sigma_A$ (respectively see Theorem 4.1.4 and Proposition 4.1.5). Fix an $\alpha \in (L, \alpha_{\text{sup}})$. For each $n \in \mathbb{N}$, let

$$X_\alpha^n := \left\{ x = (x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots) \in \Sigma_A : \frac{\sum_{i=0}^{n-1} f(\sigma^i(x))}{n} \geq \alpha \right\}.$$

Our rate function R will be composed of a conditional variational principle and a weight function. Hence, we define a function whose values are conditional variational principles. Define the function I as

$$I(\gamma) := \sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, \mathrm{d}\eta + h(\eta) : \int f \, \mathrm{d}\eta \geq \gamma \right\}$$

for each $\gamma \in (\alpha, \alpha_{\text{sup}})$. We will define our weight function $p(\alpha)$. First, we need to define the following value and function. Let

$$p_{\text{inf}} := \frac{\alpha - L}{\alpha_{\text{sup}} - L}$$

Define the function β as

$$\beta(p, \alpha) = \frac{\alpha - (1-p)L}{p} \tag{1.5.1}$$

for each $p \in (p_{\text{inf}}, 1]$. Consider the values $p(\alpha) \in (0,1)$ and $\beta(p(\alpha), \alpha) := \beta(\alpha) \in (\alpha, \alpha_{\text{sup}})$ such that

$$\beta(\alpha) = \frac{\alpha - (1 - p(\alpha))L}{p(\alpha)} \tag{1.5.2}$$

and

$$\max_{p_{\text{inf}} := \frac{\alpha - L}{\alpha_{\text{sup}} - L} \leq p \leq 1} pI(\beta(p, \alpha)) = p(\alpha) I(\beta(\alpha)). \tag{1.5.3}$$

Finally, we state our large deviation principle.

Theorem 1.5.3. *Fix* $\lambda \in \left(\frac{1}{2}, 1\right)$. *Recall the map* T_λ *given by Equation* (4.1.1)*, the shift space* (Σ_A, σ)*, and the coding map* $\pi : \Sigma_A \to (0,1]$. *Let* $\phi_\lambda := -\log|T_\lambda^{'} \circ \pi|$. *Consider* $\bar{N} := (N,N,N,...) \in \Sigma_A$ *for each* $N \in \mathbb{N}$. *Assume that* $f : \Sigma_A \to \mathbb{R}$ *is a locally Hölder potential such that* $\lim_{N\to\infty} f(\bar{N}) \in (-\infty, \infty)$. *Recall that*

$$L := \lim_{N\to\infty} f(\bar{N}) \text{ and } \alpha_{\sup} := \max_{\nu \in M_\sigma(\Sigma_A)} \left\{ \int f \, \mathrm{d}\nu \right\}.$$

Fix $\alpha \in \left(L, \alpha_{\sup}\right)$. *Then, there exists a function* R*, defined by* $p(\alpha) \in (p_{\inf}, 1]$ *and* $\beta(\alpha) \in (\alpha, \alpha_{\sup})$ *(see Equations* (1.5.1), (1.5.2)*, and eq:youthirdpalphadefiningtwo) such that*

$$\lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) = R(\alpha) = p(\alpha) \left(\sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, \mathrm{d}\eta + h(\eta) : \int f \, \mathrm{d}\eta \geq \gamma(\alpha) \right\} \right) < 0.$$

Furthermore,

$$\lim_{\alpha \to L^+} R(\alpha) = 0.$$

To prove our large deviation principle (see Theorem 1.5.3) in Chapter 4, we will create subsets of X_α^n. We will motivate the construction of these subsets by using the transient behaviour of $m-$typical sequences and the Birkhoff average of these sequences. For a full analysis of the methods used to prove this large deviation principle, see the introduction to Chapter 4.

At the end of Chapter 4 (see Section 4.10), we briefly discuss the method for forming a large deviation principle when $0 < \lambda \leq \frac{1}{2}$. This method involves using the Poincaré recurrence theorem for our reference measure and an inducing scheme. Now, we will state background knowledge for Chapters 3-4.

Chapter 2

Background

In addition to the references cited throughout this chapter, we have used Guckenheimer and Holmes [GH13], Sarig [Sar09], Mauldin [Mau95], Hasselblatt and Katok [HK02] and [HK03], Boyarsky and Gora [BG12], Dajani and Dirkskin [DD08], and Brin [BS02] to form definitions and exposition in Chapter 2.

2.1 Dynamical Systems

We define concepts and state results from dynamical systems.

2.1.1 Topological Dynamical Systems

In this thesis, we will be concentrating on discrete-time, topological dynamical systems.

Definition 2.1.1. *A* discrete-time, topological dynamical system *is made of a topological space X and a continuous map $T : X \to X$.*

Given the metric d, our topological dynamical system is the triple (X, d, T). Two topological dynamical systems can be related through a topological conjugacy.

Definition 2.1.2. *Let $T : X \to X$ and $S : Y \to Y$ be discrete-time, topological dynamical systems. A* topological semiconjugacy *from S to T is a surjective continuous map $U : Y \to X$ such that $T \circ U = U \circ S$. If U is a homeomorphism, it is called a* topological conjugacy*, and T and S are said to be* topologically conjugate or isomorphic.

Often, there will exist a topological conjugacy (up to a countable number of points) between a Markov map $T : X \to X$ and the left shift (see Proposition 2.2.9). Our results are on dynamical systems that have the following property.

Definition 2.1.3. *A discrete-time, topological dynamical system $T : X \to X$ is* topologically mixing *if for any two non-empty open sets $U, V \subset X$, there exists a $M > 0$ such that $T^m(U) \cap V \neq \emptyset$ for each $m \geq M$.*

2.2 Ergodic Theory

Now, we define concepts from ergodic theory.

2.2.1 Measure Preserving Systems

Our results are on measure preserving dynamical systems.

Definition 2.2.1. *Suppose that (X, B, m) is a probability space and the transformation $T : X \to X$ is measurable. Then, T is* measure-preserving *if $m(T^{-1}(B_1)) = m(B_1)$ for all $B_1 \in B$. Then, we will also call m an a $T-$*invariant measure.

We provide a result that characterises whether or not a measure is invariant.

Proposition 2.2.2. *Let (X, B, m) be a probability space and the transformation $T : X \to X$ be measurable. Let S be a semi-algebra that generates B. If $T^{-1}(A) \in B$ for each $A \in S$ and $m(T^{-1}(A)) = m(A)$, then m is a $T-$invariant measure.*

Proof. See Theorem 1.1 of Walters [Wal00]. ■

We consider the various measures that a topological dynamical system (X, d, T) can take. Denote the set $M(X)$ as the space of all Borel probability measures. Note that this space is convex and if the space X is compact, then $M(X)$ is compact (see Theorem 6.5 of Walters [Wal00]). For each continuous transformation $T : X \to X$, we define the subset $M(X, T)$ of $M(X)$ as

$$M(X, T) := \{\mu \in M(X) : \mu \circ T^{-1} = \mu\}.$$

We provide a theorem by Walters that lists properties of $M(X, T)$.

Theorem 2.2.3. *If T is a continuous transformation of the compact metric space X, then*

1. *$M(X, T)$ is a compact subset of $M(X)$*
2. *$M(X, T)$ is convex.*

Proof. See Theorem 6.10 of Walters [Wal00]. ■

We can also generate members of $M(X, T)$.

Proposition 2.2.4. *Let $T : X \to X$ be a continuous map on a compact metric space X. If $\{\sigma_n\}_{n=1}^{\infty}$ is a sequence in $M(X)$ and we consider the sequence $\{\mu_n\}_{n=1}^{\infty}$ by $\mu_n = \left(\frac{1}{n}\right)\sum_{i=0}^{n-1} \sigma_n \circ T^{-1}$, then any limit point μ of $\{\mu_n\}$ is a member of $M(X,T)$. (Such limits exist by the compactness of $M(X)$.)*

Proof. See Theorem 6.9 of Walters [Wal00]. ■

2.2.2 The Poincaré Recurrence Theorem

Invariant measures and the following type of set are closely related.

Definition 2.2.5. *Let $I \subset R$ be an interval and take the map $T : I \to I$. A set $A \subset I$ is* recurrent *if there exists a sequence n_k such that $T^{n_k}(x) \in A$ for each $x \in A$.*

This definition leads to the statement of Poincaré's recurrence theorem (which resulted from his work on the three body problem). In summary, take an invariant measure m on a compact space X, a measure preserving transformation T, and a positive measure set $E \subset X$. The orbit of an $m-$typical point $x \in E$ enters E infinitely often.

Theorem 2.2.6. *Let $T : X \to X$ be a measure preserving transformation of a probability space $(X, \mathscr{B}, m)$. Let $E \in \mathscr{B}$ with $m(E) > 0$. Then, almost all points of E return infinitely often to E under positive iteration by T (i.e., there exists $F \subset E$ with $m(F) = m(E)$ such that for each $x \in F$ there is a sequence $n_1 < n_2 < n_3 < ...$ of natural numbers with $T^{n_i}(x) \in F$ for each i).*

Proof. See Theorem 1.4 of Walters [Wal00]. ■

We now give specific examples of measure-preserving transformations.

2.2.3 Markov Partitions and Symbolic Dynamics

Connected to topological mixing is the notion of a Markov partition. For certain topological dynamical systems, it is possible to find such a partition. Now, we will define the terms Markov map and Markov partition on an interval. In this thesis, we will find results for the following type of map.

Definition 2.2.7. *Let $X = [a,b]$ and $T : X \to X$. Let $\mathscr{R}$ be a partition of X given by the point $a = a_0 < a_1 < \cdots < a_n = b$. For $i = 1,...,n$, let $R_i = (a_{i-1}, a_i)$ and denote the restriction of T to R_i by T_i. If T_i is a homeomorphism from R_i onto some connected union of intervals of $\mathscr{R}$, i.e., some interval $(a_{j(i)}, a_{k(i)})$, then T is said to be Markov. The partition $\mathscr{R} = \{R_i\}_{i=1}^{n}$ is referred to as a* Markov partition with respect to *T. If, furthermore, T is linear and $|T'(x)| > 0$ on each R_i, we say T is a piecewise linear Markov map.*

Let $X \subset \mathbb{R}$ be an interval. Consider the Markov map $T : X \to X$ with partition $\mathscr{R} = \{R_1, ..., R_m\}$. If $T(R_i) = X$ for all $1 \le i \le m$, we will call T a fully-branched Markov map. The preceding definition can also be modified for Markov maps with countable partitions.

Shift spaces are closely related to these maps (see Definition 2.2.8 and Proposition 2.2.9).

Definition 2.2.8. *Assume that $X \subset \mathbb{R}$ is an interval. Let $\mathscr{R} = \{R_1, ..., R_m\}$ be a Markov partition for X consisting of closed intervals. Define the $m \times m$ matrix $A = (A_{i,j})$ by*

$$A_{i,j} := \begin{cases} 0 & \text{if } Int(R_i) \cap T^{-1}(Int(R_j)) = \emptyset \\ 1 & \text{if } Int(R_i) \cap T^{-1}(Int(R_j)) \neq \emptyset. \end{cases} \tag{2.2.1}$$

Define $\Sigma_{A,m}$ to be the set of infinite sequences $\boldsymbol{a} = \{a_i\}_{i=1}^{\infty}$, $a_i \in \{1, ..., m\}$, satisfying the property $A_{a_i, a_{i+1}} = 1$ for all $i \in \mathbb{N}$. The shift map σ of such infinite sequences is given by $\sigma(\boldsymbol{a}) = \boldsymbol{b}$ when $b_i = a_{i+1}$. Clearly, $\sigma(\Sigma_{A,m}) = \Sigma_{A,m}$. The set $\Sigma_{A,m}$ together with the shift map σ is called the finite subshift of finite type with transition matrix A.

Note that the finite subshift of finite type $\Sigma_{A,m}$ is compact.

Proposition 2.2.9. *Let X be a compact locally maximal hyperbolic set and an interval and take a Markov map T with partition $\mathscr{R} = \{R_1, ..., R_m\}$ of sufficiently small diameter. There exists a conjugacy (up to a countable number of points) $\pi : \Sigma_{A,m} \to X$ that is injective on $\pi^{-1}(X') = X \setminus \cup_{i \in \mathbb{Z}} T^i(\partial^s \mathscr{R} \cup \partial^u \mathscr{R})$ and $\partial^s \mathscr{R} := \cup_{R \in \mathscr{R}} \partial^s R$ and $\partial^u \mathscr{R} := \cup_{R \in \mathscr{R}} \partial^u R$.*

Proof. See Theorem 18.7.4 of Hasselblatt and Katok [KH96]. ∎

There are also maps $T : X \to X$ that have countable Markov partitions. In this case, there would be a countable subshift of finite type (which is not compact) that we will denote by Σ_A. We will refer to Markov maps with finite (or countable) partitions as finitely-branched (or countably-branched) Markov maps. Throughout, suppose that (X, B, m) is a probability space, and the transformation $T : X \to X$ is measure-preserving.

A Markov partition can be represented symbolically as a Markov shift space. This is defined by Sarig [Sar99] as follows.

Let $\mathbb{N}$ be our countable state space. Each element of the partition is uniquely represented by a natural number. The matrix $A = (a_{i,j})$ is called a *topological transition matrix* if for all $d \in \mathbb{N}$, there exists $i, j \in \mathbb{N}$ such that $a_{d,i} = a_{j,d} = 1$. If such a matrix A exists, we define the countable state Markov shift, Σ_A, as

$$\Sigma_A := \{x \in \mathbb{N}^{\mathbb{N}} : a_{x_i, x_{i+1}} = 1 \text{ for every } i \ge 1\}.$$

Take $\sigma : \Sigma_A \to \Sigma_A$ to be the left shift. For Markov maps T, there exists a coding map $\pi : \Sigma_A \to X$.

There are many different metrics one can take on the shift space Σ_A. The following metric is often used on countable Markov shifts. Consider the metric d defined by

$$d(\boldsymbol{x},\boldsymbol{y}) := \left(\frac{1}{2}\right)^{n(\boldsymbol{x},\boldsymbol{y})}$$

such that $n(\boldsymbol{x},\boldsymbol{y}) = \inf\{n \in \mathbb{N} : x_i \neq y_i\}$ for each pair $\boldsymbol{x},\boldsymbol{y} \in \Sigma_A$. The left shift σ is continuous with respect to d, so (Σ_A, d, σ) is a topological dynamical system. We also define the topology for our countable state Markov shift Σ_A.

Definition 2.2.10. *Given $x_1,...,x_n$ symbols in $\mathbb{N}$, define a* cylinder set *in Σ_A as*

$$[x_1,...,x_n] = \{y \in \Sigma_A : y_i = x_i \text{ for } 1 \leq i \leq n\}.$$

Often, we will use a measure called a Bernoulli measure (see Page 59 of Choe [Cho06]) on Σ_A. Let $p_i \geq 0$ for each $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} p_i = 1$. A Bernoulli measure is a probability measure μ such that

$$\mu([x_1,...,x_n]) = p_{x_1} \cdots p_{x_n}$$

for each cylinder $[x_1,...,x_n] \subset \Sigma_A$. By Proposition 2.2.2, μ is invariant. Suppose that each $p_i > 0$ for each $i \in \mathbb{N}$. Then, we will later find that σ is topologically mixing (see Lemma 2.2.19). We will later define other types of measures, such as Gibbs and equilibrium measures.

We will also need to define an additional property, called ergodicity, for our transformations and measures.

2.2.4 Ergodic Measures

First, we define the notion of an ergodic transformation.

Definition 2.2.11. *Let (X, B, m) be a probability space. A measure-preserving transformation T of (X, B, m) is called* ergodic *if the only members B_1 of B, such that $T^{-1}(B_1) = B_1$, satisfy $m(B_1) = 0$ or $m(B_1) = 1$. We can also call m an* ergodic measure *if it satisfies the preceding equation for a given transformation T.*

Given the topological dynamical system (X, d, T), denote $E(X,T)$ as the set of ergodic measures in $M(X,T)$. We have the following characterisations for ergodic measures.

Theorem 2.2.12. *If T is a continuous transformation of the compact metric space X, then*

1. *μ is an extreme point of $M(X,T)$ iff T is an ergodic measure-preserving transformation of (X, B, μ) (i.e. the set of extreme points is $E(X,T)$)*

2. *If $\mu, m \in M(X,T)$ are both ergodic and $m \neq \mu$ then they are mutually singular.*

Proof. See Theorem 6.10 of Walters [Wal00]. ■

We also provide a characterisation for each Borel measure μ in terms of ergodic measures (see Page 153 of Walters [Wal00] for the following statement).

Theorem 2.2.13. *Let $T : X \to X$ be a continuous map on a compact metrisable space X. For each $\mu \in M(X,T)$, there exists a unique measure γ on the Borel subsets of the compact metrisable space $M(X,T)$, such that $\gamma(E(X,T)) = 1$ and for all $f \in C(X)$,*

$$\int_X f(x)\,\mathrm{d}\mu(x) = \int_{E(X,T)} \left(\int_X f(x)\mathrm{d}m(x) \right) \mathrm{d}\gamma(m).$$

We take $\mu = \int_{E(X,T)} m\,\mathrm{d}\gamma(m)$ and call this the ergodic decomposition *of μ.*

Proof. The result follows from the Choquet representation theorem (Theorem 3.1.11 of [PU10]) and Theorem 2.2.12. ■

In the preceding theorem, taking a map T on a compact metrisable space X is key. If X is neither compact nor metrisable, this theorem does not hold.

2.2.5 Strong Mixing

The following result helps us determine whether or not a measure is ergodic.

Theorem 2.2.14. *Let (X,B,m) be a probability space and let $T : X \to X$ be a measure-preserving transformation. Then T is ergodic iff for all $B_1, B_2 \in B$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}B_1 \cap B_2) = m(B_1)m(B_2).$$

Proof. See Corollary 1.14.2 of Walters [Wal00]. ■

This result connects to the notion of strong mixing.

Definition 2.2.15. *Let T be a measure-preserving transformation of a probability space (X,B,m). T is* strong mixing *if for all $B_1, B_2 \in B$,*

$$\lim_{n\to\infty} m(T^{-i}B_1 \cap B_2) = m(B_1)m(B_2).$$

We can call m strong mixing if T is strong mixing.

Proposition 2.2.16. *Let T be a measure-preserving transformation of a probability space (X,B,m). If m is strong mixing, then it is ergodic.*

Proof. The result follows from Theorem 2.2.14 and Definition 2.2.15. ■

Rather than using the whole Borel sigma algebra B to check whether a measure is ergodic, it is enough to simply check this property on the generating semi-algebra S for B.

Theorem 2.2.17. *Let (X,B,m) be a measure space and let S be a semi-algebra that generates B. Let $T: X \to X$ be a measure-preserving transformation. Then, T is ergodic iff for all $B_1, B_2 \subset S$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}B_1 \cap B_2) = m(B_1)m(B_2).$$

Proof. See Theorem 1.17 of Walters [Wal00]. ■

These mixing properties are similar to the notion of topological mixing. There are instances in which strong mixing yields topological mixing. To find out more about this, we define the concept of support.

Definition 2.2.18. *Let (X,B,m) be a measure and topological space. Take N as the union of all open $U \subset X$ such that $m(U) > 0$. Then, $m(N) = 1$ and N is called the support of m. We will write $supp(m) = N$.*

The following lemma states that strong mixing implies topological mixing.

Lemma 2.2.19. *Assume that (X,d,T) be a dynamical system, take $m \in M(X,T)$ such that $supp(m) = X$, and let $T: X \to X$ be a measure-preserving transformation with respect to m. If T is strong mixing, then it is topologically mixing.*

Proof. Choose arbitrary $B_1, B_2 \subset X$. We find that

$$\lim_{n\to\infty} m(T^{-n}(B_1) \cap B_2) = m(B_1)m(B_2)$$

because T is strong mixing. Then, for each $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $N \geq M$

$$\left| m(B_1)m(B_2) - m(T^{-N}(B_1) \cap B_2) \right| \leq \varepsilon.$$

Because $\text{supp}(m) = X$, $m(B_1) > 0$ and $m(B_2) > 0$. Combined, this gives us that $m(T^{-N}(B_1) \cap B_2) > 0$. Hence, there exists an $N \in \mathbb{N}$ such that $T^{-N}(B_1) \cap B_2 \neq \emptyset$. ■

2.2.6 The Birkhoff Ergodic Theorem

For measure preserving transformations T, we have a convergence law that forms the basis of ergodic theory. This law is related to the notions of time and space averages (see Page 35 of Walters [Wal00]).

Consider the dynamical system (X,d,T) and function $\phi: X \to \mathbb{R}$. For each $k \in \mathbb{N}$, take the sum

$$S_k\phi(x) := \sum_{i=0}^{k-1} \phi \circ T^i(x)$$

for each $x \in X$.

The following theorem, called the Birkhoff Ergodic Theorem, is an important result in ergodic theory.

Theorem 2.2.20. *Suppose $T:(X,B,m)\to(X,B,m)$ is a measure-preserving map (where we allow (X,B,m) to be $\sigma-$finite) and $\phi\in\mathscr{L}^1(m)$. Then*

$$\lim_{k\to\infty}\frac{1}{k}S_k\phi(x)=\phi^*(x)$$

for $m-$a.e. $x\in X$ with $\phi^*\in\mathscr{L}^1(m)$. Also, $\phi^*\circ T=\phi^*$ a.e. Furthermore, if $m(X)<\infty$ and T is ergodic, then

$$\phi^*(x)=\int\phi\,\mathrm{d}m.$$

Proof. See Theorem 1.14 of Walters [Wal00]. ■

We will sometimes need uniform rather than pointwise convergence when we use results, such as the Birkhoff Ergodic Theorem. Hence, we provide a result, called Egoroff's Theorem, that ensures that sets, with non-zero measure, in which uniform convergence occurs exist.

Theorem 2.2.21. *Suppose that $\mu(X)<\infty$ and $f_1,f_2,\ldots$ and f are measurable complex-valued functions on X such that $f_n\to f$ a.e. Then for every $\varepsilon>0$, there exists $E\subset X$ such that $\mu(E)<\varepsilon$ and $f_n\to f$ uniformly on $E^{\complement}$.*

Proof. See Theorem 2.33 of Folland [Fol13]. ■

Now, we will define a function, called entropy, on ergodic measures.

2.2.7 Entropy

Entropy is known to be a conjugacy and isomorphism invariant (see Theorem 4.11 of Walters [Wal00] for a discussion). For instance, consider the doubling map $T:[0,1]\to[0,1]$ defined by $T(x)=2x \bmod 1$ and the left shift $\sigma:\Sigma_2\to\Sigma_2$ such that $\Sigma_2:=\{1,2\}^{\mathbb{N}}$. T is conjugate to σ because there exists a coding map $\pi:\Sigma_2\to[0,1]$ such that $T\circ\pi=\pi\circ\sigma$. We will later find that the measure theoretic entropies of T and σ are equal. A less trivial example is the Gauss map $G:[0,1]\setminus\mathbb{Q}\to[0,1]\setminus\mathbb{Q}$ defined by $G(x)=\frac{1}{x} \bmod 1$. The map G is conjugate to σ (except on a countable number of points) and the homeomorphism defining this conjugacy (up to a countable number of points) is $\pi:\mathbb{N}^{\mathbb{N}}\to[0,1]\setminus\mathbb{Q}$, the continued fraction map. Likewise, we will find that the measure theoretic entropies of G and σ are equal. However, this might not always been the case for other maps and their respective shift spaces. We will later use this conjugacy invariance and entropy's relation to Hausdorff dimension and the growth rate of Birkhoff sums.

We will mainly discuss measure-theoretic entropy.

2.2.8 Partitions of a Probability Space

Consider a probability space (X,B,m).

Definition 2.2.22. *A* partition *of (X,B,m) is a disjoint set of $x\in B$ whose union is X.*

When calculating the entropy of a function, we will want to refine a given partition.

Definition 2.2.23. *Suppose ξ and η are finite partitions of (X,B,m). If each element of ξ is a union of elements in η, we denote this by $\xi \le \eta$. η is called the* refinement *of ξ.*

One way of further refining a partition is called joining.

Definition 2.2.24. *Let $\xi = \{A_1,...,A_n\}$, $\eta = \{C_1,...,C_k\}$ be two finite partitions of (X,B,m). Their* join *is*

$$\xi \vee \eta := \{A_i \cap C_j : 1 \le i \le n, 1 \le j \le k\}.$$

We define a partition associated to a measure-preserving transformation T.

Definition 2.2.25. *Suppose $T : X \to X$ is a measure-preserving transformation. If $\xi = \{A_1,...,A_k\}$, then $T^{-n}\xi$ denotes the partition $\{T^{-n}A_1,...,T^{-n}A_k\}$.*

We find a few relations between joins, refinements, and the preceding definition: $T^{-n}(\xi \vee \eta) = T^{-n}\xi \vee T^{-n}\eta$ and if $\xi \le \eta$, then $T^{-n}\xi \le T^{-n}\eta$.

2.2.9 The Entropy of a Partition

Again, let us consider the partition $\xi = \{A_1,...,A_n\}$ of X. We define the information function as given by Equation (4.3.2) of Hasselblatt and Katok [KH96].

Definition 2.2.26. *Consider a probability space (X,B,m) and take an $x \in X$. Let $\xi(x)$ be the element of ξ that contains x. We define the* information function *$I_\xi : X \to \mathbb{R}$ by*

$$I_\xi(x) := -\log m(\xi(x)).$$

We now define the entropy of a partition.

Definition 2.2.27. *Consider a probability space (X,B,m) and our partition ξ. The* entropy of a partition *is given by*

$$H_\mu(\xi) := \int I_m(\xi)\,\mathrm{d}m = -\sum_{i=1}^{n} m(A_i)\log m(A_i).$$

2.2.10 The Entropy of a Measure Preserving Map

Consider a probability space (X,B,m) and our partition $\xi = \{A_1,...,A_k\}$. We will create a join from our partition ξ and a measure-preserving map $T : X \to X$.

Definition 2.2.28. *Consider a probability space (X,B,μ), our partition ξ, and a measure-preserving transformation $T : X \to X$. We define the* joint partition *by*

$$\xi_{-n}^{T} := \vee_{i=0}^{n-1} T^{-i}\xi := \xi \vee T^{-1}\xi \vee T^{-2}\xi \vee \cdots \vee T^{-(n-1)}\xi = \{A_{i_0} \cap T^{-1}A_{i_1} \cap T^{-2}A_{i_2} \cdots \cap T^{-(n-1)}A_{i_{n-1}} : 1 \le i_j \le k\}.$$

This leads to a result about the entropies of joint partitions.

Proposition 2.2.29. *Consider a probability space* (X,B,μ)*, our partition* ξ*, and a measure-preserving transformation* $T: X \to X$*. Then,*

$$\lim_{n\to\infty} \frac{1}{n} H_\mu \left(\vee_{i=0}^{n-1} T^{-i} \xi \right)$$

exists.

Proof. See Corollary 4.9.1 Walters [Wal00]. ■

This proposition justifies defining the entropy of a transformation with respect to a partition ξ.

Definition 2.2.30. *Consider a probability space* (X,B,μ)*, our partition* ξ*, and a measure-preserving transformation* $T: X \to X$*. The* metric entropy of T *relative to the partition* ξ *is*

$$h_\mu(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu \left(\vee_{i=0}^{n-1} T^{-i} \xi \right) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu \left(\vee_{i=0}^{n-1} T^{-i} \xi \right).$$

If T is a Markov map, it tells us the speed in which the measure of the joint partition generated by a Markov partition or a cylinder set decreases. From this, we define the entropy of a transformation.

Definition 2.2.31. *Consider a probability space* (X,B,μ)*, our partition* ξ*, and a measure-preserving transformation* $T: X \to X$*. The* entropy of T *with respect to* μ *is*

$$h_\mu(T) := \sup\{h_\mu(T,\xi) : \xi \subset B \textit{ is a finite or countable partition and } h_\mu(T,\xi) < \infty\}.$$

2.2.11 Methods To Calculate Entropy

There are more practical ways to calculate the entropy of T.

Theorem 2.2.32. *Consider a probability space* (X,B,μ)*, our partition* ξ*, and a measure-preserving transformation* $T: X \to X$*. Take a sequence of finite or countable partitions* $\{\xi_n\} \subset B$ *of* X *satisfying*

$$\xi_1 \le \xi_2 \le \cdots \le \xi_{n-1} \le \xi_n \le \cdots$$

such that $\vee_{n=-\infty}^{\infty} \xi_n = B$ *(mod* μ*). We then have that*

$$h_\mu(T) = \lim_{n\to\infty} h_\mu(T,\xi_n).$$

Proof. See Theorem 4.22 of Walters [Wal00]. ■

Another method for calculating $h_\mu(T)$ is selecting partitions that maximise $h_\mu(T,\xi)$. This result is called the Kolmogorov-Sinai Theorem.

Theorem 2.2.33. *Consider a probability space (X,B,μ) and an invertible measure-preserving transformation $T:X\to X$. Let A be a finite sub-algebra of B such that $\vee_{n=-\infty}^{\infty}T^nA = B$ (mod μ). Then, $h_\mu(T) = h_\mu(T,\xi)$.*

Proof. See Theorem 4.17 of Walters [Wal00]. ■

In general, we can consider the entropy of a transformation for various invariant measures. This leads us to the notion of the entropy map.

2.2.12 The Entropy Map

Now, we will define the entropy map. Assume that (X,d) is a compact metric space and $T:X\to X$ is continuous. Denote $M(X,T)$ as the space of $T-$invariant measures. Walters [Wal00] states that this space is a non-empty, convex set that is compact in the weak* topology (see Theorem 2.2.3).

Definition 2.2.34. *The* entropy map *of the continuous transformation $T:X\to X$ is the map $\mu\mapsto h_\mu(T)$ defined on $M(X,T)$ and its range is* $[0,\infty]$.

The entropy map has the following property.

Theorem 2.2.35. *Let $T:X\to X$ be a continuous map of a compact space. The entropy map of T is affine, i.e., if $\mu,m\in M(X,T)$ and $p\in[0,1]$ then $h_{p\mu+(1-p)m}(T) = ph_\mu(T)+(1-p)h_m(T)$.*

Proof. See Theorem 8.1 of Walters [Wal00]. ■

Before stating another property of the entropy map, we provide the following definition.

Definition 2.2.36. *A homeomorphism T of a compact metric space (X,d) is called* expansive *if there exists a $\delta>0$ with the property that if $x\neq y$ then there exists an $n\in\mathbb{N}\cup\{0\}$ such that $d(T^n(x),T^n(y))>\delta$.*

The entropy map of these transformations has the following property.

Theorem 2.2.37. *When $T:X\to X$ is an expansive homeomorphism of a compact metric space, the entropy map of T is upper semi-continuous, i.e., for each $\mu\in M(X,T)$ and $\varepsilon>0$, there exists a neighbourhood U of μ in $M(X,T)$ such that each $m\in U$ satisfies $h_m(T)<h_\mu(T)+\varepsilon$.*

Proof. See Theorem 8.2 of Walters [Wal00]. ■

The upper semi-continuity of the entropy map might not hold for measures on non-compact spaces. For instance, consider measures on a countable full Markov shift. We remark that one can also define conditional entropy (see Definition 4.3.2 of Hasselblatt and Katok [KH96]). Finally, we finish with an necessary result used in our various results' proofs.

2.2.13 The Shannon McMillan Breiman Theorem

Consider the measure-preserving dynamical system (X,μ,B,T) (i.e. T is a measure-preserving map) and ξ a finite or countable partition. The Shannon McMillan Breiman Theorem states that, on average, the measure of the joint partition $\vee_{i=0}^{n-1} T^{-i}\xi$ decreases exponentially at a rate given by the entropy $h_\mu(T,\xi)$. See Page 93 of [Wal00] and Theorem 7 in Chapter 2 of [Par04] for the following statement of the Shannon McMillan Breiman Theorem.

Theorem 2.2.38. *Consider a probability space (X,B,m), a partition ξ satisfying $\vee_{i=0}^{n-1} T^{-i}\xi = B$ (mod m), and an ergodic measure-preserving transformation $T : X \to X$. Then,*

$$\lim_{n\to\infty} -\frac{1}{n} \log m(\vee_{i=0}^{n-1} T^{-i}\xi \cap C) = h_m(T,\xi)$$

for $m-$typical $x \in X$.

Proof. See Theorem 7 in Chapter 2 of [Par04]. ∎

We will now give examples.

2.2.14 The Entropy for The Left Shift on Σ_2, Σ_k, and $\mathbb{N}^{\mathbb{N}}$

Take $k \in \mathbb{N} \setminus 1$. Let us consider the expanding map $T : [0,1] \to [0,1]$ such that $T(x) = kx$ mod 1.

From this map, we get the shift space $\Sigma_k := \{1,...,k\}^{\mathbb{N}}$. Give each symbol $i \in \{1,...,k\}$ a weight p_i such that $\sum_{i=1}^{k} p_i = 1$ and define the measure μ on each cylinder $[x_1,...,x_n] \subset \Sigma_k$ by

$$\mu([x_1,...,x_n]) = p_{x_1} \cdots p_{x_n}.$$

The measure μ is a Bernoulli measure on Σ_k. The left shift $\sigma : \Sigma_k \to \Sigma_k$ is conjugate (up to a countable number of points) to the map T.

Proposition 2.2.39. *The Bernoulli shift Σ_k has entropy $h_\mu(\sigma) = -\sum_{i=1}^{k} p_i \log p_i$.*

Proof. See Theorem 4.26 of Walters [Wal00]. ∎

Consider the left shift $\sigma : \Sigma_2 \to \Sigma_2$ and the Bernoulli measure μ defined by $p_1 = p_2 = \frac{1}{2}$. Then, the Bernoulli shift on 2 symbols has entropy $h_\mu(\sigma) = \log(2)$. Consider the doubling map $T(x) = 2x$ mod 1, the coding map $\pi : \Sigma_2 \to [0,1]$ (satisfying $T \circ \pi = \pi \circ \sigma$), and the measure $\bar{\mu}$. Because there is a conjugacy (up to a countable number of points) between T and σ, $h_\mu(\sigma) = \log 2 = h_{\bar{\mu}}(T)$. Proposition 2.2.39 can be extended to the full countable Markov shift $\mathbb{N}^{\mathbb{N}}$, its left shift $\sigma : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, and Bernoulli measure $\mu : \mathbb{N}^{\mathbb{N}} \to (0,1)$. In this case, $h_\mu(\sigma) = -\sum_{i=1}^{\infty} p_i \log p_i$.

We give an example of an expanding map with a countable Markov partition that is conjugate (up to a countable number of points) to the left shift $\sigma : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. Fix $\lambda \in (0,1)$. Consider the expanding map $S_\lambda : (0,1] \to (0,1]$:

$$S_\lambda(x) := \begin{cases} \frac{x-\lambda}{1-\lambda} & \text{if } x \in (\lambda, 1] \\ \frac{x-\lambda^n}{\lambda^{n-1}(1-\lambda)} & \text{if } x \in (\lambda^n, \lambda^{n-1}] \text{ for each } n \geq 2. \end{cases} \tag{2.2.2}$$

S_λ has a Markov partition $\{S_1, ...\}$ such that $S_n := (\lambda^n, \lambda^{n-1}]$ for each $n \in \mathbb{N}$. This map is a modification of the expanding map T_λ, originally studied by [SV97] and given by Equation (4.1.1), used in Chapter 4 and 5. Its countable Markov partition $\{R_1, R_2, ...\}$ is given by $R_1 := (\lambda, 1], ..., R_i := \left(\lambda^i, \lambda^{i-1}\right], ...$ From S_λ, we get the shift space $(\mathbb{N}^{\mathbb{N}}, \sigma)$ because the expanding map has a conjugacy (up to a countable number of points) with σ (i.e., there exists a coding map $\pi : \mathbb{N}^{\mathbb{N}} \to (0,1]$ such that $S_\lambda \circ \pi = \pi \circ \sigma$ and $\pi^{-1}(R_n) = [n]$). Consider the Bernoulli measure ν such that

1. $\nu([i]) = \lambda^{i-1}(1-\lambda)$ and
2. $\nu([x_1, ..., x_n]) = \nu([x_1]) \cdots \nu([x_n])$.

Then,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu([x_1, ..., x_n]) = -\sum_{i=1}^{\infty} \lambda^{i-1}(1-\lambda) \log[\lambda^{i-1}(1-\lambda)] = h_\nu(\sigma) = h_{\nu\circ\pi^{-1}}(T_\lambda)$$

by the Shannon McMillan Breiman Theorem because the Markov partition $\xi := \{R_1, R_2, ...\}$ maximises $h_\mu(T, \xi)$ (hence, $h_\mu(T,\xi) = h_\mu(T)$ by Theorem 2.2.33) and T_λ and σ are conjugate.

2.2.15 Topological Entropy

Topological entropy is closely related to measure theoretic entropy. First, we define the following.

Assume that (X, d) is a compact metric space and $T : X \to X$ is a continuous transformation. Define the distance function d_n for all $x, y \in X$ by

$$d_n(x, y) := \max\{d(T^k(x), T^k(y)) : 0 \le k \le n-1\}$$

for each $n \in \mathbb{N}$

Definition 2.2.40. *Assuming that $\varepsilon > 0$, a finite set $E \subset X$ is called $(n, \varepsilon)-$separated if $d_n(x, y) > \varepsilon$ for every $x, y \in E$ such that $x \neq y$.*

We are now ready to define topological entropy. This quantity gives the exponential rate in which the orbit of any two points $x, y \in X$ diverge from each other.

Definition 2.2.41. *We define the* topological entropy *of the transformation T by*

$$h(T) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n, \varepsilon),$$

where $N(n, \varepsilon)$ is the largest cardinality of all $(n, \varepsilon)-$separated sets.

We also have the variational principle for entropy.

Theorem 2.2.42. *Let T be a continuous map on a compact metric space X. The topological entropy of T is given by*

$$h(T) = \sup_{\mu \in M(X,T)} h_\mu(T).$$

Proof. See Theorem 8.6 of Walters [Wal00]. ■

From this point forward, we will denote $h_\mu(T)$ as $h(\mu)$ for each fixed map T. Topological entropy is closely related to the idea of pressure, which is defined on the next section.

2.3 Thermodynamic Formalism

To understand the dynamic and ergodic theoretic properties of our expanding Markov maps, we will use the thermodynamic formalism for both finite state and countable state Markov shifts. While thermodynamic formalism is an area of ergodic theory, we devote a section to it because of its importance to our results.

2.3.1 Thermodynamic Formalism for Finite State Markov Shifts

We will consider an expanding, Markov map $T : X \to X$ with a finite Markov partition R. This partition is an example of a basic set and this map is an example of an Axiom A diffeomorphism (see Pages 47-48 of Bowen [BC75] for these definitions). Thus, the map T can be modelled by a finite state Markov shift Σ_A. We will state results on the thermodynamic formalism for finite state Markov shifts.

Fix any $k \in \mathbb{N}$. Take $K := \{1,...,k\}$ as our countable state space and $A = (a_{l,m})_{K\times K}$ as our transition matrix of zeros and ones. Note that A can be represented by a directed graph. Denote $\Sigma_k := \{1,...,k\}^{\mathbb{N}}$. We let

$$\Sigma_A := \{x \in \{1,...,k\}^{\mathbb{N}} : a_{x_i,x_{i+1}} = 1 \text{ for every } i \geq 1\}.$$

For simplicity, we will be giving definitions and theorems on the compact space Σ_k. We take $\sigma : \Sigma_k \to \Sigma_k$ to be the standard left shift. We now define the topology for our finite state Markov shift Σ_k.

Definition 2.3.1. *Given $x_1,...,x_m \in K$, define a* cylinder set *in Σ_k as*

$$[x_1,...,x_m] = \{y \in \Sigma_k : y_i = x_i \text{ for } 1 \leq i \leq n\}.$$

These cylinder sets form the topology for Σ_A. Note that Σ_k is topologically mixing. Recall that Σ_k is topologically mixing if for any two non-empty $U, V \subset \Sigma_k$, there exists an $N \in \mathbb{N}$ such that $\sigma^m U \cap V \neq \emptyset$ for all $m \geq N$. Given $x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in \Sigma_k$, define the function

$$n(x,y) := \sup\{i \in \mathbb{N} : x_m = y_m \text{ for all } 1 \leq m \leq i\}.$$

We define the metric d on Σ_k as

$$d(x,y) = \left(\frac{1}{2}\right)^{n(x,y)}$$

for any $x, y \in \Sigma_k$.

A potential is a function that maps Σ_k to $\mathbb{R}$. The shift space Σ_k is compact, which is an important condition for the following concepts. We first introduce pressure.

Definition 2.3.2. *For a continuous potential $\phi:\Sigma_k \to \mathbb{R}$, the limit*

$$\mathscr{P}_\sigma(\phi) := \lim_{k\to\infty} \frac{1}{k} \log \sum_{[x_1,\dots,x_k]\subset\Sigma} \exp \sup_{y\in[x_1,\dots,x_k]} \left(\sum_{i=0}^{k-1} \phi(\sigma^i(y)) \right)$$

is called the pressure *of ϕ.*

We state the variational principle.

Theorem 2.3.3. *Let $\sigma:\Sigma_k \to \Sigma_k$ be the left shift and $\phi:\Sigma_k \to \mathbb{R}$ be a continuous function. Then, $\mathscr{P}_\sigma(\phi) = \sup_{\mu\in M(\Sigma_k,\sigma)} \left\{\int \phi \, \mathrm{d}\mu + h(\mu)\right\}$.*

Proof. See Theorem 20.2.4 of Hasselblatt and Katok [KH96]. ■

Then,

$$\mathscr{P}_\sigma(\phi) := \mathscr{P}(\phi) = \sup_{\mu\in M(\Sigma_k,\sigma)} \left\{ \int \phi \, \mathrm{d}\mu + h(\mu) \right\}$$

by Theorem 2.3.3.

Often, a potential will have a measure that achieves the supremum given by the variational principle in Theorem 2.3.3.

Definition 2.3.4. *Let $\sigma:\Sigma_k \to \Sigma_k$ be the left shift and $\phi:\Sigma_k \to \mathbb{R}$ be a continuous function. A measure $\mu \in M(\Sigma_k,\sigma)$ that satisfies*

$$\mathscr{P}(\phi) = \int \phi \, \mathrm{d}\mu + h(\mu),$$

is called an equilibrium state.

If T is expansive, we have a result about the existence of equilibrium states.

Proposition 2.3.5. *Let $T:\Sigma_k \to \Sigma_k$ be a continuous map, σ be the left shift, and $\phi:\Sigma_k \to \mathbb{R}$ be a continuous function. Then, there is an equilibrium state for ϕ.*

Proof. See Theorem 20.2.10 of Hasselblatt and Katok [KH96]. ■

We recall the definition of Hölder continuity.

Definition 2.3.6. *A potential $\phi:\Sigma_k \to \mathbb{R}$ is Hölder continuous if there are constants $a,\theta > 0$ so that*

$$|\phi(x) - \phi(y)| \le a d(x,y)^\theta.$$

We will define another type of measure that relates the measure of each cylinder set in Σ_k to the Birkhoff sum of each point in the cylinder. Let $M(\Sigma_k,\sigma) := M_\sigma(\Sigma_k)$.

Definition 2.3.7. *Suppose $\phi : \Sigma_k \to \mathbb{R}$ is Hölder continuous. Then, the measure $\mu \in M_\sigma(\Sigma_k)$ is a* Gibbs measure *for ϕ if there exist constants $c_1 > 0, c_2 > 0$, and P such that*

$$c_1 \leq \frac{\mu([x_1,\ldots,x_n])}{\exp\left(-Pn + \sum_{i=0}^{n-1} \phi(\sigma^i(x))\right)} \leq c_2$$

for every $x = (x_1,\ldots,x_n,x_{n+1},\ldots) \in \Sigma_k$.

The constant $P = P(\phi)$ (see Theorem 1.22 of Bowen [BC75]). We give a result about the existence of Gibbs and equilibrium measures for potentials on Σ_k.

Proposition 2.3.8. *Suppose $\phi : \Sigma_k \to \mathbb{R}$ is Hölder. Then, there exists a unique Gibbs measure μ for ϕ. Furthermore, μ is the equilibrium state for ϕ.*

Proof. See Theorems 1.4 and 1.22 of Bowen [BC75]. ■

We will finish with a result about the analyticity of pressure. First, we give a definition. Take $\mathscr{H}$ be the class of Hölder functions.

Definition 2.3.9. *Two functions $\phi : \Sigma_k \to \mathbb{R}$ and $\psi : \Sigma_k \to \mathbb{R}$ are* cohomologous *in a class $\mathscr{H}$ if there exists a function $u : \Sigma \to \mathbb{R}$ in the class $\mathscr{H}$ such that*

$$\phi - \psi = u - u \circ \sigma.$$

Otherwise, the potential ϕ is non-cohomologous to ψ.

Now, we state a result (stated as Theorem 5.12 in Barreira [BWBO08] but proven by Ruelle [Rue04]) about the analyticity of pressure. Fix an $\varepsilon \in (0,1]$. Take $\phi, \psi : \Sigma_k \to \mathbb{R}$ be Hölder with exponent ε.

Theorem 2.3.10. *Take the Hölder potentials $\phi, \psi : \Sigma_k \to \mathbb{R}$. The following properties hold:*

1. *the map $t \mapsto \mathscr{P}(\phi + t\psi)$ is analytic;*
2. *the unique equilibrium measure μ for ϕ is ergodic and*
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{P}(\phi + t\psi)|_{t=0} = \int \psi \, \mathrm{d}\mu;$$
3. *the Hölder potentials ϕ, ψ have the same equilibrium state if and only if $\phi - \psi$ is cohomologous to a constant;*
4. *for each $t \in \mathbb{R}$,*
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathscr{P}(\phi + t\psi) \geq 0,$$
with equality if and only if ψ is cohomologous to a constant.

We will find that the existence of Gibbs and equilibrium states for potentials on countable Markov shifts is not guaranteed. The introduction of extra combinatorial conditions is necessary.

2.3.2 Thermodynamic Formalism for Countable State Markov Shifts

Expanding maps that have countable Markov partitions can be modelled by a countable Markov shift (Σ, σ). We state the necessary theory, developed by Sarig [Sar99], called the thermodynamic formalism for countable state Markov shifts.

Let $\mathbb{N}$ be our countable state space and $A = (a_{l,m})_{\mathbb{N}\times\mathbb{N}}$ be our transition matrix of zeros and ones. Note that A can be represented by a directed graph. We let

$$\Sigma_A := \{x \in \mathbb{N}^{\mathbb{N}} : a_{x_i,x_{i+1}} = 1 \text{ for every } i \geq 1\}.$$

We take $\sigma : \Sigma_A \to \Sigma_A$ to be the standard left shift. We now define the topology for our countable state Markov shift Σ_A.

Definition 2.3.11. *Given $x_1, ..., x_m \in \mathbb{N}$, define a* cylinder set *in Σ_A as*

$$[x_1, ..., x_m] = \{y \in \Sigma_A : y_i = x_i \text{ for } 1 \leq i \leq n\}.$$

These cylinder sets form the topology for Σ_A. Two important assumptions for our countable Markov shift are defined below.

Definition 2.3.12. *The shift space Σ_A satisfies the* big images and pre-images (BIP) property *if there is a finite set $\{c_1, c_2, ..., c_m\}$ from our alphabet $\mathbb{N}$ such that for each $d \in \mathbb{N}$, there are $i, j \in \{1, ..., m\}$ such that*

$$a_{c_i,d}a_{d,c_j} = 1.$$

Definition 2.3.13. *$\sigma : \Sigma_A \to \Sigma_A$ is said to be* topologically mixing *if for all $a, b \in \mathbb{N}$, there exists $N_{ab} \in \mathbb{N}$ such that for all $n > N_{ab}$,*

$$[a] \cap \sigma^{-n}[b] \neq \emptyset.$$

We give Sarig's definition of topological mixing (see Page 286 of Sarig [Sar01b]) because $1-$cylinders are open sets that generate the topology of Σ_A. Because Σ_A is a non-compact shift space, Sarig [Sar03] proves that the presence of topological mixing and the BIP property for Σ_A are two of the necessary conditions for ϕ to have a Gibbs measure (see Theorem 2.3.28).

Mauldin and Urbański [MU03] define a property, connected to topological mixing, called finite irreducibility. Denote E^* as the set of all finite subwords of Σ_A.

Definition 2.3.14. *The matrix A is* finitely irreducible *if there exists a finite set $\Lambda \subset E^*$ such that for all $i, j \in \mathbb{N}$ there exists a word $\omega \in \Lambda$ for which $i\omega j \in E^*$.*

If A is finitely irreducible, we will state that Σ_A is finitely irreducible. We provide a useful result from Page 5 of Mauldin and Urbański [MU03] that connects finite irreducibility to topological mixing and the BIP property.

Lemma 2.3.15. *If $\sigma : \Sigma_A \to \Sigma_A$ is topologically mixing, then Σ_A is finitely irreducible if and only if it satisfies the big images and pre-images property.*

Denote Σ_A as Σ. Since Σ satisfies the big images and pre-images property and is topological mixing, it is finitely irreducible. Now, we give a definition for functions on Σ.

Definition 2.3.16. *Let $\psi : \Sigma \to \mathbb{R}$. The k-th* variation *of ψ is given by*

$$V_k(\psi) := \sup_{[x_1,\ldots,x_k]\subset\Sigma_A} \sup_{x,y\in[x_1,\ldots,x_k]} |\psi(x)-\psi(y)|.$$

Definition 2.3.17. *The function $\psi : \Sigma \to \mathbb{R}$ is said to be* locally Hölder continuous *if there exists $C > 0$ and $\theta \in (0,1)$ such that for each $k \in \mathbb{N}$,*

$$V_k(\psi) \le C\theta^k.$$

For our results, we take the following definition of a locally constant potential.

Definition 2.3.18. *A* locally constant potential *is a function $\tilde{\psi} : \Sigma \to \mathbb{R}$ satisfying the following. There exists an $n \in \mathbb{N}$ such that for each $[x_1,\ldots,x_n] \subset \Sigma$,*

$$\tilde{\psi}(x) = c$$

for a $c := c([x_1,\ldots,x_n]) \in \mathbb{R}$ and any $x \in [x_1,..,x_n]$.

This proposition relates locally Hölder potentials to locally constant potentials. It follows from the definition of local Hölder continuity.

Proposition 2.3.19. *For each locally Hölder potential $\psi : \Sigma \to \mathbb{R}$, there exists a locally constant potential $\tilde{\psi}$ satisfying the following. For each $[x_1,\ldots,x_n] \subset \Sigma$, there exists a sufficiently small $\varepsilon > 0$ such that*

$$\sup_{x\in[x_1,\ldots,x_n]} |\psi(x) - \tilde{\psi}(x)| \le \varepsilon. \tag{2.3.1}$$

We will state that $\tilde{\psi}$ approximates ψ if $\tilde{\psi}$ satisfies Equation (2.3.1).

We give another definition involving locally Hölder potentials.

Definition 2.3.20. *Two functions $\phi : \Sigma \to \mathbb{R}$ and $\psi : \Sigma \to \mathbb{R}$ are* cohomologous *in a class $\mathcal{H}$ if there exists a function $u : \Sigma \to \mathbb{R}$ in the class $\mathcal{H}$ such that*

$$\phi - \psi = u - u \circ \sigma.$$

Otherwise, the potential ϕ is non-cohomologous to ψ.

Now, we define an important function of ψ, the topological pressure of ψ, used in the proofs of Theorems 3.1.5 and 3.1.6. Mauldin and Urbański [MU03] define this type of pressure in their text.

Definition 2.3.21. *Let $\psi : \Sigma \to \mathbb{R}$ be locally Hölder. We define the* topological pressure *as*

$$\mathscr{P}_{MU}(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{[x_1,\dots,x_n]\subset\Sigma} \exp \sup_{y\in[x_1,\dots,x_n]} \left(\sum_{i=0}^{n-1} \psi(\sigma^i(y)) \right).$$

We also define another form of pressure.

Definition 2.3.22. *Let $\psi : \Sigma \to \mathbb{R}$ be locally Hölder. We define the* Gurevich pressure *as*

$$\mathscr{P}_G(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp \left(\sum_{i=0}^{n-1} \psi(\sigma^i(x)) \right) \mathbb{1}_{[i]}(x)$$

such that $\mathbb{1}_{[i]}$ is the indicator function on the cylinder $[i]$ with $i \in \mathbb{N}$.

Because Σ is topologically mixing, the Gurevich pressure of ψ does not depend on the choice of $x_1 \in \mathbb{N}$ (see Theorem 1 of Sarig [Sar99]). We also introduce variational pressure.

Definition 2.3.23. *Let Σ be topologically mixing and $\phi : \Sigma \to \mathbb{R}$ be locally Hölder such that* $\sup\psi < \infty$. *Let $M_\sigma(\Sigma)$ be the set of $\sigma-$invariant measures. Then, the* variational pressure *is*

$$\mathscr{P}(\psi) = \sup_{\mu\in M_\sigma(\Sigma)} \left\{ \int \psi \, \mathrm{d}\mu + h(\mu) \right\}. \tag{2.3.2}$$

A measure $\mu \in M_\sigma(\Sigma)$ satisfying Equation (2.3.2) *is called an* equilibrium state.

We find the following result by Sarig.

Proposition 2.3.24. *Assume that Σ is topologically mixing and $\psi : \Sigma \to \mathbb{R}$ be locally Hölder. Then,*

$$\mathscr{P}(\psi) = \mathscr{P}_G(\psi).$$

Proof. See Theorem 3 of Sarig [Sar99]. ■

We combine the preceding results on pressure.

Proposition 2.3.25. *Let Σ be topologically mixing and satisfy the BIP property. Assume that $\psi : \Sigma \to \mathbb{R}$ is locally Hölder such that* $\sup\psi < \infty$. *Let $M_\sigma(\Sigma)$ be the set of $\sigma-$invariant measures. Then,*

$$\mathscr{P}(\psi) = \mathscr{P}_{MU}(\psi) = \mathscr{P}_G(\psi).$$

Proof. This result follows from Proposition 2.3.24 and Theorem 2.1.8 of Mauldin and Urbański [MU03]. ■

We remark that Iommi, Jordan, and Todd [IJT15] (Page 8, Theorem 2.10) proved that $\sup\psi < \infty$ is an unnecessary condition for the previous proposition. We will provide a result that gives another way to calculate the topological pressure of a potential. Let ϕ and ψ be locally Hölder. We can approximate the topological pressure of a potential $q\phi - t\psi$ (for fixed $q, t \in \mathbb{R}$) on the shift

Σ by considering the restriction of the potential to a compact, invariant subset $K \subset \Sigma$. For such a set K, denote

$$\mathscr{P}_K(q\phi - t\psi) := \mathscr{P}((q\phi - t\psi)|_K)$$

as the topological pressure of the restriction of $q\phi - t\psi$ to K. Sarig [Sar99] (Page 1570, Theorem 2) and Mauldin and Urbański [MU03] (Page 8, Theorem 2.1.5) have proven the following proposition.

Proposition 2.3.26. *Assume that Σ is topologically mixing and satisfies the BIP property. Let $\phi : \Sigma \to \mathbb{R}$ and $\psi : \Sigma \to \mathbb{R}$ be locally Hölder and $q, t \in \mathbb{R}$ be fixed. If $\mathscr{K} := \{K \subset \Sigma : K$ compact and σ-invariant, $K \neq \emptyset\}$, then*

$$\mathscr{P}(q\phi - t\psi) = \sup_{K \in \mathscr{K}} \mathscr{P}_K(q\phi - t\psi).$$

We will simply use the term pressure for the function $\mathscr{P}$ instead of topological pressure because of Proposition 2.3.25. Compact subsets of our countable Markov shift Σ include finite state Markov shifts. We will later use a nested sequence of these shifts to approximate the topological pressure of a potential on Σ.

Definition 2.3.27. *A probability measure μ is said to be a* Gibbs measure *for the potential $\phi : \Sigma \to \mathbb{R}$ if there exist two constants $M, P > 0$ such that, for each cylinder $[x_1, x_2, ..., x_m]$ and every $x \in [x_1, x_2, ..., x_m]$,*

$$\frac{1}{M} \le \frac{\mu([x_1, x_2, ..., x_m])}{\exp\left(-mP + \sum_{j=0}^{m-1} \phi\left(\sigma^j(x)\right)\right)} \le M. \tag{2.3.3}$$

In fact, Mauldin and Urbański [MU03] (Page 13, Proposition 2.2.2) proved that $P = \mathscr{P}(\phi)$. We give a few results about the existence of Gibbs and equilibrium states for potentials.

Theorem 2.3.28. *Let (Σ, σ) be topologically mixing and $\gamma : \Sigma \to \mathbb{R}$ be locally Hölder. Then, γ has an invariant ergodic Gibbs state if and only if the transition matrix A has the BIP property and $\mathscr{P}(\gamma) < \infty$.*

Proof. See Theorem 1 of Sarig [Sar03]. ■

We provide a similar theorem by Mauldin and Urbański.

Theorem 2.3.29. *Let (Σ, σ) be topologically mixing and satisfy the BIP property and $\gamma : \Sigma \to \mathbb{R}$ be locally Hölder. Then, if γ has an invariant Gibbs state, it is unique and ergodic.*

Proof. See Theorem 2.2.4 of Mauldin and Urbański [MU03]. ■

The following result combines Theorems 2.3.28 and 2.3.29.

Theorem 2.3.30. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}$ be locally Hölder such that $\mathscr{P}(\phi) < \infty$. Then, there exists a unique invariant, ergodic Gibbs state μ for ϕ. If this measure also satisfies $\int \phi \, \mathrm{d}\mu > -\infty$, μ is the unique equilibrium state for ϕ.*

The analyticity of the pressure function on locally Hölder potentials is not guaranteed. Hence, we provide a definition and corollary from Sarig [Sar03] (Page 1756, Corollary 4).

Consider the following definition.

Definition 2.3.31. *Assume that* (Σ,σ) *satisfies the BIP property and is topologically mixing. Let* $\phi:\Sigma\to\mathbb{R}$ *be a locally Hölder potential. Denote* $Dir(\phi)$ *as the collection of all* $\psi:\Sigma\to\mathbb{R}$ *such that there exists* $C_\phi>0, r\in(0,1)$ *and* $\varepsilon>0$ *with*

1. $V_m(\phi)<C_\phi r^m$ *for all* $m\geq 1$
2. $\mathscr{P}(\phi+t\psi)<\infty$ *for all* $t\in(-\varepsilon,\varepsilon)$.

We have the following result that states that $t\mapsto\mathscr{P}(\phi+t\psi)$ is real analytic for each $t\in(-\varepsilon,\varepsilon)$.

Proposition 2.3.32. *Assume that* (Σ,σ) *satisfies the BIP property and is topologically mixing. Let* $\phi:\Sigma\to\mathbb{R}$ *be a locally Hölder potential and* $\psi\in Dir(\phi)$. *Then, there exists* $\varepsilon>0$ *for which* $t\mapsto\mathscr{P}(\phi+t\psi)$ *is real analytic on* $(-\varepsilon,\varepsilon)$.

Proof. See Corollary 4 of Sarig [Sar03]. ∎

Hence, we can take the derivative of pressure as follows.

Theorem 2.3.33. *Assume that* (Σ,σ) *satisfies the BIP property and is topologically mixing. Let* $\phi:\Sigma\to\mathbb{R}$ *be a locally Hölder potential and* $\psi\in Dir(\phi)$. *Take* μ *as the Gibbs measure for* ϕ. *Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{P}(\phi+t\psi)|_{t=0}=\int\psi\,\mathrm{d}\mu.$$

Proof. See Proposition 2.6.13 of Mauldin and Urbański [MU03]. ∎

To understand thermodynamic properties of a map, it is sometimes necessary to use an inducing scheme.

2.4 Inducing Scheme

Although we will not be using an inducing scheme for any of the main theorems of Chapters 3-4, we will be discussing its use in the introduction to Chapter 4. We will define the following inducing scheme obtained from Sarig's paper [Sar01b], Pages 291-292 because his setting is similar to ours in Chapter 4. Consider an ergodic probability preserving, expanding, Markov map $(X,B,\bar{\mu},T)$ and its shift space (Σ,σ) with measure $\mu=\bar{\mu}\circ\pi$.

Definition 2.4.1. *For each measurable, partition set* $A\subset\Sigma$ *with positive measure, we define the* induced transformation $\sigma_A:A\to A$ *by*

$$\sigma_A(x):=\sigma^{\phi_A(x)}(x)$$

where

$$\phi_A(x) := \inf\{n \geq 1 : \sigma^n(x) \in A\}.$$

Define the conditional measure μ_A by

$$\mu_A(E) := \frac{\mu(A \cap E)}{\mu(A)}$$

for each $E \subset A$. One often takes a measure μ such that $\mu(A) = 1$.

We form an inducing scheme, given by Page 292 of Sarig [Sar01b], as follows. This scheme is used when $\mu-$typical $x \in \Sigma$ returns to A infinitely often and is closely connected to the Poincaré Recurrence Theorem.

Let

$$\overline{S} := \{[\underline{a}] \subset A : A \text{ appears only once in } \underline{a} \text{ and } [\underline{a}, A] \neq \emptyset\}$$

and $\overline{\Sigma} := \overline{S}^{\mathbb{N}}$. By the Poincaré Recurrence Theorem, $\mu_A(\cup\overline{S}) = 1$.

Take ρ to be the natural projection:

$$\rho([\underline{a}_1], [\underline{a}_2], \ldots) = (\underline{a}_1, \underline{a}_2, \ldots)$$

such that $a_i \in \overline{S}$. Consider the measure $\tilde{\mu} := \mu_A \circ \rho^{-1}$ on $\overline{\Sigma}$. Let the left shift on $\bar{\Sigma}$ as $\bar{\sigma} = \sigma_A$. There is a conjugacy (up to a countable number of points) between $T|_A \circ \pi$ and $\bar{\sigma}$.

Consider a locally Hölder potential $f : \Sigma \to \mathbb{R}$. Define the induced potential $\overline{f} : \overline{\Sigma} \to \mathbb{R}$ as

$$\overline{f}(x) = \sum_{i=0}^{\phi_A(x)-1} f(\sigma^i(\rho(x)))$$

for each $x \in \overline{\Sigma}$. Then, if $f \in \mathscr{L}^1(\mu)$,

$$\frac{\int \overline{f} \, \mathrm{d}\tilde{\mu}}{\int \overline{\phi}_A \, \mathrm{d}\tilde{\mu}} = \int f \, \mathrm{d}\mu_A.$$

Abrahmov's result (see [Abr59]) for the entropy of $\tilde{\mu}$ also applies.

On Page 292 of [Sar01b], Sarig cites Section 1.5 of Aaronson [Aar97] to form the preceding inducing scheme. Consider an ergodic probability preserving, expanding, Markov map $(X, B, \bar{\mu}, T)$, its topologically mixing shift space (Σ, μ, σ), a measurable partition set $A \subset \Sigma$ with positive measure, and the induced map σ_A. Take the conditional measure μ_A given by

$$\mu_A(E) := \frac{\mu(A \cap E)}{\mu(A)}$$

for each $E \subset A$.

We will also state a further result about induced potentials.

Proposition 2.4.2. *Consider an ergodic probability preserving, expanding, Markov map* $(X, B, \bar{\mu}, T)$ *and its topologically mixing shift space* (Σ, σ). *If* ϕ *is locally Hölder, then* $\bar{\phi}$ *is locally Hölder. Also, if* $\mathscr{P}_G(\phi) = 0$, *then* $\mathscr{P}_G(\bar{\phi}) = 0$.

Proof. See Lemma 3 of Sarig [Sar01a]. ■

Forming an inducing scheme can be useful when working with countable Markov shifts. It can help one find results in large deviations (see the introduction to Chapter 4) and multifractal analysis (see Section 5 of Iommi [Iom05]).

The study of the pressure function is connected to multifractal analysis.

2.5 Multifractal Analysis

We apply thermodynamic formalism to the study of multifractal analysis. Our exposition for this area of dimension theory is fairly brief, so see the various sources mentioned in this section for more thorough analyses. In multifractal analysis, one decomposes a fractal into level sets and studies the size or dimension of these sets. We first define the notion of Hausdorff dimension.

2.5.1 Hausdorff and Local Dimension

Suppose that (X,ρ) is a metric space. In this section, we take $|U|$ to be the the diameter of each set $U \subset X$. Fix $s \in [0,\infty)$. For each $\delta > 0$, define

$$H_s^\delta(A) := \inf\left\{\sum_{i=1}^\infty |U_i|^s : A \subset \bigcup_{i=1}^\infty U_i \text{ and } |U_i| < \delta \text{ for each } i \in \mathbb{N}\right\}$$

for each $A \subset X$. H_s^δ is an outer measure, so the following limit is an outer measure if it exists:

$$H_s(A) = \lim_{\delta\to 0} H_s^\delta(A).$$

The measure H_s is called the s-dimensional Hausdorff measure.

Definition 2.5.1. *Given $A \subset X$, the* Hausdorff dimension *of A is the unique value t such that*

$$H_s(A) := \begin{cases} \infty \textit{ if } 0 \le s < t \\ 0 \textit{ if } t < s < \infty. \end{cases} \tag{2.5.1}$$

We denote $\dim_H(A)$ as the Hausdorff dimension of A.

Lemma 2.5.2. *If $\{A_n\}_{n\ge 1}$ is a countable family of subsets of X then*

$$\dim_H\left(\bigcup_n A_n\right) = \sup_n\{\dim_H(A_n)\}.$$

Proof. See Theorem A.2.0.11 of Mauldin and Urbański [MU03]. ■

The Hausdorff dimension of a measure can also be defined.

Definition 2.5.3. *Let μ be a Borel measure on (X,ρ). Then, the* Hausdorff dimension $\dim_H(\mu)$ *of the measure μ is defined as*

$$\dim_H(\mu) := \inf\{\dim_H(Y) : \mu(X \setminus Y) = 0\}.$$

We define another, closely related, type of dimension called the pointwise or local dimension.

Definition 2.5.4. *Let μ be a Borel measure on the metric space (X,ρ). For each $x \in X$, the* upper pointwise dimension *of μ at x is*

$$\overline{d}_\mu(x) := \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

and the lower pointwise dimension *of μ at x is*

$$\underline{d}_\mu(x) := \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

If the two are equal, their common value is the pointwise or local dimension $d_\mu(x)$ *of μ at x.*

We have a mass distribution principle (see Theorem A2.0.16 of Mauldin and Urbański [MU03]).

Proposition 2.5.5. *Suppose that μ is a Borel probability measure on $\mathbb{R}^n, n \geq 1$.*

1. *If there exists $\theta_1 \in [0,\infty)$ such that for μ–a.e. $x \in \mathbb{R}^n$*

   $$\underline{d}_\mu(x) \geq \theta_1$$

   then $\dim_H(\mu) \geq \theta_1$.

2. *If there exists $\theta_1 \in [0,\infty)$ such that for μ–a.e. $x \in \mathbb{R}^n$*

   $$\underline{d}_\mu(x) \leq \theta_2$$

   then $\dim_H(\mu) \leq \theta_2$.

Proof. See Theorems 8.6.3 and 8.6.4 of Przytycki and Urbański [PU10]. ∎

We have a near converse called Frostman's lemma.

Proposition 2.5.6. *Let $E \subset \mathbb{R}^n$ be a nonempty Borel set and let s be a positive real number.*

1. *If* $\dim_H(E) > s$, *then there exists a Borel measure μ with $0 < \mu(E) < \infty$ and $\underline{d}_\mu(x) \geq s$ for all $x \in E$.*

2. *If* $\dim_H(E) < s$, *then there exists a Borel measure μ with $0 < \mu(E) < \infty$ and $\underline{d}_\mu(x) \leq s$ for all $x \in E$.*

Proof. See Proposition 3.6.1 of Edgar [Edg07]. ∎

2.5.2 Multifractal Spectrum

Suppose we have a countable Markov shift (Σ, σ) that is topologically mixing and satisfies the BIP property. We consider a locally Hölder potential $\phi : \Sigma \to \mathbb{R}$ with Gibbs state μ. A standard type of multifractal spectrum (also called the dimension spectrum) will be defined shortly. First, we restate the definition of local dimension on our shift space. Let $\alpha \geq 0$.

Definition 2.5.7. *A sequence $x = (x_1, x_2, ..., x_m, ...) \in \Sigma$ has* local dimension *α if*

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha.$$

Recall that the local dimension is also called the pointwise dimension. Let

$$\alpha_{\inf} := \inf \left\{ \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} : x \in \Sigma \right\} \text{ and}$$

$$\alpha_{\sup} := \sup \left\{ \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} : x \in \Sigma \right\}.$$

We will later give another definition for $\alpha_{\inf}$ and $\alpha_{\sup}$ (see Lemma 3.5.5) by using thermodynamic formalism. For each $\alpha \in [\alpha_{\inf}, \alpha_{\sup}]$, we consider the set

$$X_\alpha := \left\{ x \in \Sigma : \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}. \tag{2.5.2}$$

We denote

$$X'' := \left\{ x \in \Sigma : \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ does not exist} \right\}.$$

Hence, the shift space

$$\Sigma = \biguplus_{\alpha \in [\alpha_{\inf}, \alpha_{\sup}]} X_\alpha \biguplus X''.$$

We will analyse the following function, which gives the Hausdorff dimension of each X_α.

Definition 2.5.8. *Consider the set X_α given by Equation* (2.5.2). *The* multifractal spectrum *is the function f_μ defined by*

$$\alpha \mapsto \dim_H(X_\alpha) \tag{2.5.3}$$

such that $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$.

Note that we can construct the multifractal spectrum similarly on a finite state Markov shift Σ_n. Dimension theory is also closely linked to iterated function systems.

2.6 Iterated Function Systems

Assume that (X,d) is a compact metric space.

Definition 2.6.1. *An* iterated function system *on X (indexed by a finite or countable alphabet E) is a finite or countable list $(f_e)_{e\in E}$ of functions $f_e : X \to X$.*

Sometimes,the term iterated function system will instead be called an IFS. We will be working with contractions.

Definition 2.6.2. *An iterated function system is* uniformly contracting or hyperbolic *if and only if there is a Lipschitz constant $r<1$ such that*

$$d(f_e(x), f_e(y)) \leq r d(x,y)$$

for all $x,y \in S$ and all $e \in E$.

Consider the shift space $\Sigma := E^{\mathbb{N}}$. Given $x=(x_1,...,x_n,...)\in\Sigma$, let $f_x := \lim_{n\to\infty} f_{x_1}\circ\cdots\circ f_{x_n}$. There is a natural coding map $\pi : \Sigma \to X$ such that

$$\pi(x) = f_x(E).$$

Take a finite, uniformly contracting iterated function system $(f_e)_{e\in E}$. Then, there exists a set K, called an attractor, that satisfies

$$K = \bigcup_{e\in E} f_e(K)$$

(see Theorem 9.1 of Falconer [Fal04]).

We will consider iterated function systems in which $f_i(K)\cap f_j(K)=\emptyset$ for each $i,j\in E$ such that $i\neq j$. Our examples in Chapter 3 include infinite interval iterated function systems (see Definition 2.6.3). First, we define the notion of an iterated function system.

Definition 2.6.3. *Let E be an finite (infinite) indexing set. A* finite (infinite) iterated function system *is a collection of $\{f_i : i \in E\}$ of differentiable, uniform contractions, $f_i : I \to I$.*

We will consider conformal iterated function systems (see Page 71-72 of Mauldin and Urbański [MU03]). Our settings in Chapter 3 are the interval and the shift, so a discussion of conformal IFSs is unnecessary.

2.6.1 Dimension of a Conformal Attractor

This subsection unites thermodynamic formalism with iterated function systems. Take a finite iterated function system $(f_e)_{e\in E}$. Consider the attractor $A\subset X$:

$$A = \bigcup_{e\in E} f_e(A)$$

and its pre-image $\pi^{-1}(A) = K$. Consider the finite state Markov shift $\Sigma := E^{\mathbb{N}}$ generated from this IFS. Let us define a potential $\psi : \Sigma \to \mathbb{R}$ by $\psi(x) = -\log|f'_{x_1} \circ \pi|$ and if we have a measure $\mu \in M_\sigma(\Sigma)$, the Lyapunov exponent, $\lambda(\mu) := \int \psi \, d\mu$.

This result, called Bowen's equation, was proven by Ruelle [Rue04].

Theorem 2.6.4. *Consider a finite interval iterated function system $\{f_e\}_{e\in E}$ and the finite state Markov shift $\Sigma := E^{\mathbb{N}}$. Then, there exists a unique $s \geq 0$ such that $\mathscr{P}(-s\psi) = 0$ and $s = \dim_H(K)$. Furthermore,*

$$\dim_H(K) = \sup_{\mu \in M_\sigma(\Sigma)} \frac{h(\mu)}{\lambda(\mu)}.$$

Proof. See Theorem 20.1 of Pesin [Pes08]. ■

A similar result can be given for the attractor of a countable interval iterated function system and the recurrent set in a countable Markov shift (see Theorem 3.1 of Iommi [Iom05]). There are similar results in multifractal analysis.

2.7 Multifractal Analysis for Measures on Finite and Countable State Shifts

We first discuss results on the multifractal analysis for equilibrium measures on finite state Markov shifts. Recall the definition of the multifractal spectrum in Subsection 2.5.2.

2.7.1 Multifractal Analysis for Equilibrium States on Finite State Markov Shifts

We defined the multifractal spectrum for a Gibbs measure on a countable Markov shift. However, we can consider the Gibbs state for a Hölder continuous function on a finite state Markov shift and then, provide a similar construction for the multifractal spectrum.

Consider an expanding, finitely branched (that is, it has a finite Markov partition) Markov map $M : I \to I$, such that $I \subset \mathbb{R}$, that is topologically mixing. This map is modelled by a finite state Markov shift Σ_A that is topologically mixing and has a coding map $\pi : \Sigma_A \to I$. Assume that $\phi : \Sigma_A \to \mathbb{R}$ is a Hölder continuous function such that $\mathscr{P}(\phi) = 0$ and consider the a Hölder continuous potential $\psi : \Sigma_A \to \mathbb{R}$ such that $\psi := \log|M' \circ \pi|$.

Definition 2.7.1. *Consider the function $T : \mathbb{R} \to \mathbb{R}$ given implicitly by the expression*

$$\mathscr{P}(-T(q)\psi + q\phi) = 0$$

for each $q \in \mathbb{R}$.

We will be using T in our analysis of the multifractal spectrum's analyticity, so we provide a result about the analyticity of T. See Theorem 6.12 of Barreira [BWBO08] for its statement and their proof (which we provide below).

Proposition 2.7.2. *Consider an expanding, finitely branched (that is, it has a finite Markov partition) Markov map $M : I \to I$, such that $I \subset \mathbb{R}$, that is topologically mixing. This map is modelled by a finite state, topologically mixing Markov shift Σ_A and has a coding map $\pi : \Sigma_A \to I$. Take the Hölder potentials ϕ and ψ that we previously defined. If μ is the equilibrium measure of a Hölder continuous function $\phi : \Sigma_A \to \mathbb{R}$ with $\mathscr{P}(\phi) = 0$, then*

1. *the map*
$$(t,q) \mapsto \mathscr{P}(-t\psi + q\phi)$$
is analytic and
2. *the function T is analytic.*

Proof. Because $t\psi + q\phi$ is Hölder for every pair of $t,q \in \mathbb{R}$,

$$(t,q) \mapsto \mathscr{P}(-t\psi + q\phi)$$

is analytic by Theorem 2.3.10. As found by Barreira [BWBO08],

$$\frac{\partial}{\partial t}\mathscr{P}(-t\psi + q\phi) = -\int_{\Sigma_A} \psi \, \mathrm{d}\mu_{t,q} = -\int_{\Sigma_A} \log|M' \circ \pi| \, \mathrm{d}\mu_{t,q}$$

such that $\mu_{t,q}$ is the unique equilibrium state for $-t\psi + q\phi$. Then, since M is expanding,

$$\frac{\partial}{\partial t}\mathscr{P}(-t\psi + q\phi) < 0$$

for each $t,q \in \mathbb{R}$. Hence, by the implicit function theorem, the function T is analytic and well-defined. ∎

Pesin and Weiss [PW97] consider the multifractal spectrum of a Gibbs measure. We will state their result, as given in Barreira. Let $\alpha : \mathbb{R} \to \mathbb{R}_0^+$ be given by $\alpha(q) = -T'(q)$. Let ν be the equilibrium measure of ϕ and μ_q be the equilibrium measure of $-T(q)\psi + q\phi$.

Now, we state Pesin and Weiss's result (also stated by Theorem 6.1.2 of Barreira [BWBO08]) on the multifractal analysis of equilibrium measures.

Theorem 2.7.3. *Consider an expanding, finitely branched (that is, it has a finite Markov partition) Markov map $M : I \to I$, such that $I \subset \mathbb{R}$, that is topologically mixing. This map is modelled by a finite state, topologically mixing Markov shift Σ_A and has a coding map $\pi : \Sigma_A \to I$. Take the Hölder potentials ϕ and ψ that we previously defined. If μ is the equilibrium measure of a Hölder continuous function $\phi : \Sigma_A \to \mathbb{R}$ with $\mathscr{P}(\phi) = 0$, then*

1. *the set $X_{\alpha(q)}$ is $\sigma-$invariant and dense for every $q \in \mathbb{R}$;*

2. *if $\nu = \mu$, then $\alpha_{\sup} - \alpha_{\inf} = 1$ and f_μ is a delta function;*

3. *if $\nu \neq \mu$, then $f_\mu : (\alpha_{\inf}, \alpha_{\sup}) \to \mathbb{R}$ is analytic and convex;*

4. *f_μ is the Legendre transform of T, that is, for each $q \in \mathbb{R}$,*

$$f_\mu(\alpha(q)) = T(q) + q\alpha(q).$$

5. *for each $q \in \mathbb{R}$, we have $\mu_q(X_{\alpha(q)}) = 1$ and*

$$\lim_{r\to 0} \frac{\log \mu_q(B(x,r))}{\log r} = T(q) + q\alpha(q)$$

for μ_q-almost every $x \in X_{\alpha(q)}$.

Proof. See Theorem 6.1.2 of Barriera [BWBO08]. ∎

Pesin and Weiss [PW97] consider the following example. They take a finite interval iterated function system given by the family of contractions $\{S_i\}_{i\in I}$, such that $S_i : [0,1] \to [0,1)$, with a finite indexing set $I := \{1,...,k\}$. Each S_i has contracting factor s_i such that $\sum_{i=1}^k s_i = 1$ and we assign each contraction a weight p_i such that $\sum_{i=1}^k p_i = 1$. We can consider the associated finite state Markov shift $\Sigma_k := I^{\mathbb{N}}$. There is a Gibbs (Bernoulli) measure μ given by

$$\mu([x_1,...,x_n]) = p_{x_1} \cdots p_{x_n}$$

for each cylinder set $[x_1,...,x_n] \subset \Sigma_k$. Then, they found that the local dimension of each $x = (x_1,...,x_n,...) \in \Sigma_k$ is given by

$$d_\mu(x) = \lim_{n\to\infty} \frac{\log \mu([x_1,...,x_n])}{\log |[x_1,...,x_n]|} = \lim_{n\to\infty} \frac{\log(p_{x_1}\cdots p_{x_n})}{\log |s_{x_1}\cdots s_{x_n}|}.$$

In this case, we take $\phi(x) = \log p_{x_1}$ and $\psi(x) = -\log s_{x_1}$ for each $x = (x_1,...,x_n,...) \in \Sigma_k$. Then, they defined the function T by the equation

$$\mathscr{P}(q\phi - T(q)\psi) = \log\left(\sum_{i=1}^k p_i^q s_i^{T(q)}\right) = 0.$$

Finally, they found that $f_\mu(\alpha) = T(q) + q\alpha$ for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$. For these α there exists a $q \in \mathbb{R}$ such that $\alpha = \alpha(q)$.

We will form similar examples (that are infinite iterated function systems) to this one in Chapter 3. We will shortly give results by Iommi [Iom05] on the multifractal analysis for Gibbs measures on countable state Markov shifts.

2.7.2 Multifractal Analysis for Gibbs Measures on Countable State Markov Shifts

Recall the definition of the multifractal spectrum given by Definition 2.5.8. Equilibrium states might not necessarily exist for a given potential on a countable Markov shift, so we use Gibbs states. Also, the multifractal spectrum for a Gibbs measure might not be analytic everywhere on its domain. This subsection concentrates on results from Iommi [Iom05] by using the setting of Chapter 3.

Consider an expanding, countably branched (that is, it has a countable Markov partition) Markov map $M : I \to I$, such that $I \subset \mathbb{R}$, that is topologically mixing and satisfies the BIP property (see Definition 2.3.12). Furthermore, this map must have a Markov partition $\{R_1, R_2, ..., R_m, ...\}$ such that $\lim_{m\to\infty} R_m = 0$ and the limit $\lim_{m\to\infty} \frac{\log|R_{x_1,...,x_m}|}{\log|R_{x_1,...,x_m,x_{m+1}}|} = 1$. This map is modelled by a countable state Markov shift Σ_A that is topologically mixing and satisfies the BIP property. Consider the coding map $\pi : \Sigma_A \to I$. Assume that $\phi : \Sigma_A \to \mathbb{R}$ is a locally Hölder continuous function such that $\mathscr{P}(\phi) = 0$ and consider the potential $\psi : \Sigma_A \to \mathbb{R}$ such that $\psi := \log|M' \circ \pi|$.

Definition 2.7.4. *Take an arbitrary cylinder* $[x_1, ..., x_m] \subset \Sigma$. *Define the diameter of* $[x_1, ..., x_m]$ *as*

$$|[x_1, ..., x_m]| := \sup_{x,y\in[x_1,...,x_m]} d(x, y)$$

for a metric d. *We consider metrics* d *such that*

$$\frac{1}{C}\prod_{j=0}^{m-1}(\exp(\psi(\sigma^j(y))))^{-1} \le |[x_1, x_2, ..., x_m]| \le C\prod_{j=0}^{m-1}(\exp(\psi(\sigma^j(y))))^{-1},$$

for every $y = (x_1, x_2, ..., x_m, y_{m+1}, ...) \in [x_1, ..., x_m]$ *and a constant* $C > 0$. *We will call our function* ψ *a* metric potential.

To understand the behaviour of the multifractal spectrum, we must analyse the pressure function $t \mapsto \mathscr{P}(q\phi - t\psi)$ for each fixed $q \in \mathbb{R}$. Hence, we define the following function.

Definition 2.7.5. *For each* $q \in \mathbb{R}$, *the* temperature function $T(q)$ *is*

$$T(q) := \inf\{t \in \mathbb{R} : \mathscr{P}(q\phi - t\psi) \le 0\}.$$

Iommi [Iom05] uses his choice of metric (given by Definition 2.7.4) and the notion of symbolic dimension to prove his results. He proves an expression for the multifractal spectrum.

Theorem 2.7.6. *Assume that* (Σ, σ) *satisfies the BIP property and is topologically mixing. Let* $\phi : \Sigma \to \mathbb{R}^-$ *be a locally Hölder potential such that* $\mathscr{P}(\phi) = 0$ *and* $\psi : \Sigma \to \mathbb{R}^+$ *be a metric potential. Assume that* ϕ *is non-cohomologous to* ψ. *Let the Gibbs state for* ϕ *be denoted by* μ. *For each* $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$,

$$f_\mu(\alpha) = \inf_{q\in\mathbb{R}}\{T(q) + q\alpha\}.$$

Proof. See Theorem 4.1 of Iommi [Iom05]. ■

We will add the assumption that $\mathscr{P}(-\psi) < \infty$ to prove that the multifractal spectrum can have phase transitions (non-analytic points). This behaviour is a contrast to the multifractal spectrum for an equilibrium state on a finite state Markov shift. Using the same techniques from thermodynamic formalism, we can also solve problems from large deviations.

2.8 Large Deviations for Hyperbolic Dynamical Systems

The following is a typical problem in large deviations in hyperbolic dynamical systems. Consider an expanding map $T : I \to I$ for an interval $I \subset \mathbb{R}$. Define the function $\phi := -\log|T'|$ and a reference measure m. Consider a function $f : I \to \mathbb{R}$ such that $f \in \mathscr{L}^1(m)$. Without loss of generality, fix an arbitrary $\alpha > \int f \,\mathrm{d}m$. Then, consider the following set:

$$X_\alpha^n := \left\{ x \in I : \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \geq \alpha \right\}$$

for each $n \in \mathbb{N}$. Typically, one aims to find a rate function R (defined shortly) such that

$$\lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) = R(\alpha) < 0. \tag{2.8.1}$$

Let $C(X,\mathbb{R})$ be the space of continuous functions from X to $\mathbb{R}$.

We give the following modified version of Definition 5 from Young's paper [You90].

Definition 2.8.1. *Given a dynamical system $f : X \to X$ with reference measure m and potential $\phi \in C(X,\mathbb{R})$, we say that $\frac{1}{n}S_n\phi$ satisfies a* large deviation principle with rate function $k : X \to [-\infty, 0]$ *if*

1. *k is upper semicontinuous;*

2. *for every open set $E \subset \mathbb{R}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log m\left(\frac{1}{n} S_n\phi \in E\right) \geq \sup\{k(s), s \in E\};$$

3. *for every closed set $E \subset \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log m\left(\frac{1}{n} S_n\phi \in E\right) \leq \sup\{k(s), s \in E\}.$$

We will provide one of Young's [You90] most important results on large deviations. Let $\varepsilon > 0$ and fix an $n \in \mathbb{N}$. Define the set

$$V(x,n,\varepsilon) = \{y \in X : d(f^i x, f^i y) < \varepsilon, 0 \leq i < n\}.$$

Consider the sets

$$V^{+} = \left\{\xi \in C(X,\mathbb{R}) : \exists C, \varepsilon > 0 \text{ s.t.} \forall x \in X \text{ and } \forall n \geq 0, m(V(x,n,\varepsilon)) \leq Ce^{-S_n\xi(x)}\right\} \text{ and}$$

$$V^{-} = \left\{\xi \in C(X,\mathbb{R}) : \exists \text{arbitrarily small } \varepsilon > 0 \text{ and } C = C(\varepsilon) \text{ s.t.} \forall x \in X \text{ and } \forall n \geq 0, m(V(x,n,\varepsilon)) \geq Ce^{-S_n\xi(x)}\right\}.$$

Before we provide this result, we need the following definition.

Definition 2.8.2. *f satisfies* specification *if for every $\theta > 0, \exists p = p(\theta) \in \mathbb{Z}^+$ s.t. given any k points $x_1,...,x_k \in X$, $n_1,...,n_n \in \mathbb{Z}^+$, and $p_1,...,p_{k-1} \geq p(\theta)$, $\exists x \in X$ s.t.*

$$d(f^i x, f^i x_1) < \theta, 0 \leq i < n_1$$

$$d(f^{n_1+p_1+i} x, f^i x_2) < \theta, 0 \leq i < n_2$$

$$\vdots$$

$$d(f^{n_1+\cdots+n_{k-1}+p_1+\cdots+p_{k-1}+i} x, f^i x_k) < \theta, 0 \leq i < n_k.$$

Let $E \subset X$. For each $n \in \mathbb{N}$, let $E_n := \left\{x \in X : \frac{1}{n}S_n\phi(x) \in E\right\}$.

Theorem 2.8.3. *Let X be a compact metric space, $f : X \to X$ be a continuous map, and m be a Borel measure. Assume $h(f) < \infty$. Then, for every $\phi \in C(X,\mathbb{R})$, $c \in \mathbb{R}$, we find the following.*

1. *Take $E = [c,\infty)$. For $\xi \in V^+$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log m(E_n) \leq \sup\{h(\nu) - \int \xi \, d\nu : \nu \in M(X,f), \int \phi \, d\nu \geq c\}.$$

2. *Take $E = (c,\infty)$. Assume f satisfies specification. Then, for $\xi \in V^-$, we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log m(E_n) \geq \sup\{h(\nu) - \int \xi \, d\nu : \nu \in M(X,f), \int \phi \, d\nu > c\}.$$

Proof. See Theorem 1 of Young [You90]. ∎

Theorem 2.8.3 leads to a large deviation principle for finite state Markov shifts Σ_A. Let $\underline{p} = (p_1,...,p_n)$ be a vector such that $\sum_{i=1}^n p_i = 1$ and $p_i > 0$ for all $1 \leq i \leq n$. Take $P = (P_{i,j})$ as an $n \times n$ stochastic matrix. Let $m \in M(\Sigma_A, \sigma)$ be a Markov measure and P be irreducible (see Pages 16 and 22 of Walters [Wal00] for both definitions).

Theorem 2.8.4. *Consider $\sigma : \Sigma_A \to \Sigma_A$ with reference measure m. Then for every $\phi \in C(\Sigma_A,\mathbb{R})$, $\frac{1}{n}S_n\phi$ has large deviation principle with rate function*

$$k(s) = -\sup\left\{h(\nu) + \int \log P_{x_0,x_1} \, d\nu(\underline{x}) : \nu \in M(\Sigma_A,\sigma), \int \phi \, d\nu = s\right\}.$$

Moreover, $k(s_0) = 0$ for a unique s_0.

Proof. See Theorem 5 of Young [You90]. ■

We provide the definition for a large deviation principle, stated on Young's paper [You90] and modified by others such as [CT17], for a dynamical system T on a non-compact space X.

Definition 2.8.5. *Given a dynamical system $T : X \to X$ with reference measure m and locally Hölder observable $f : X \to \mathbb{R}$, we say that $\frac{1}{n} S_n f$ satisfies a* large deviation principle with rate function *$k : X \to [-\infty, 0]$ if*

1. *for every open set $E \subset \mathbb{R}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log m\left(\frac{1}{n} S_n f \in E\right) \geq \sup\{k(s), s \in E\};$$

2. *for every closed set $E \subset \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log m\left(\frac{1}{n} S_n f \in E\right) \leq \sup\{k(s), s \in E\}.$$

The techniques used in large deviations can vary, so we respectively leave the discussion of our large deviations in Chapters 4 and 5.

CHAPTER 3

Phase Transitions of the Multifractal Spectrum

The results from Sections 1-7,9 come from [Dun14]. However, we have revised its content since [Dun14] was posted on ArXiv.

3.1 Introduction

See Chapter 1 for a discussion of previous work done in multifractal analysis. A standard form of multifractal analysis is on the study of the concentration of a measure on level sets. This chapter considers a problem in this type of multifractal analysis. Finitely-branched, expanding Markov maps are modelled through finite state shift spaces. Using measures on these spaces, multifractal analysts found an expression of the standard multifractal spectrum and proved that it is concave and analytic. In contrast, consider countably-branched, expanding Markov maps and their countable Markov shift spaces. Then, for measures on these spaces, the multifractal spectrum might not be analytic. Take a countable Markov shift that is topologically mixing and satisfies the big images and pre-images property or BIP property (see Definition 2.3.12). Iommi [Iom05] forms an expression for the multifractal spectrum with respect to a Gibbs measure on this space.

We consider the standard multifractal analysis for Gibbs measures on countable Markov shifts. This chapter uses Iommi's [Iom05] expression of the multifractal spectrum to analyse its non-analytic points or phase transitions. By Sarig's [Sar99] thermodynamic formalism for countable Markov shifts, we apply analyticity arguments on the pressure function on a countable shift in our analysis. From this analysis, we form Theorems 3.1.5 and 3.1.6 about the various possible phase transitions of the multifractal spectrum. We find upper and lower bounds for the number of the multifractal spectrum's phase transitions. Finally, we apply our results (see Theorems 3.1.5 and 3.1.6) on the multifractal spectrum's phase transitions to the Gauss map.

However, we provide a brief discussion (after Definition 3.9.3) on these theorems' applicability to other expanding, countably-branched Markov maps.

3.1.1 Setting

We first discuss our problem's setting. Take a countable Markov shift space Σ that is topologically mixing and satisfies the BIP property. We consider the locally Hölder potential functions $\phi : \Sigma \to \mathbb{R}^-$, such that $\mathscr{P}(\phi) = 0$ (see Definition 2.3.21 for the definition of topological pressure), and $\psi : \Sigma \to \mathbb{R}^+$ such that $\mathscr{P}(-\psi) < \infty$ (the significance of finite pressure will be explained in the discussion surrounding Equation (3.3.5)). There exists a Gibbs state μ for ϕ by Theorem 3.3.7. Our choice of metric d will depend on ψ (see Definition 3.1.3). This choice of metric will be key for our construction of the multifractal spectrum (see Definition 3.5.6 and Propositions 3.5.4 and 3.5.7). Additionally, we assume that ϕ is non-cohomologous (see Definition 2.3.20) to ψ and the following limit $\alpha_{\lim}$ exists. Consider an arbitrary allowable sequence $\hat{i} = (i, y_2, y_3, \dots) \in [i] \in \Sigma$. Take

$$\alpha_{\lim} := \lim_{i \to \infty} \frac{\phi(\hat{i})}{-\psi(\hat{i})}$$

if it exists. Our remark after Definition 3.3.5 will prove that the limit is independent of the choice of $\hat{i} \in [i]$. Theorem 3.1.5 assumes that $0 < \alpha_{\lim} < \infty$ and Theorem 3.1.6 assumes that $\alpha_{\lim} = \infty$. To ensure that it is possible for $0 < \alpha_{\lim} \le \infty$, we need to assume that $\mathscr{P}(-\psi) < \infty$ (see the discussion around Equation (3.3.5) and Lemma 3.3.4).

To prove that the the multifractal spectrum has phase transitions, we analyse the pressure function $t \mapsto \mathscr{P}(q\phi - t\psi)$ for each fixed $q \in \mathbb{R}$. Hence, we define the following two functions. Definition 4.2 of [Iom05] gives us the following definition.

Definition 3.1.1. *For each $q \in \mathbb{R}$, the* temperature function

$$T(q) := \inf\{t \in \mathbb{R} : \mathscr{P}(q\phi - t\psi) \le 0\}$$

According to Lemma 4.1 of [Iom05], $T(q)$ might not exist for $q < 0$. However, this does not affect our argument for Theorem 3.1.5. We also use this similarly defined function.

Definition 3.1.2. *For each $q \in \mathbb{R}$, the* function

$$\tilde{t}(q) := \inf\{t \in \mathbb{R} : \mathscr{P}(q\phi - t\psi) < \infty\}.$$

We also consider the value

$$t_\infty := \inf\{t \in \mathbb{R} : \mathscr{P}(-t\psi) < \infty\},$$

which is used to form an expression for $\tilde{t}(q)$ (see Proposition 3.3.6). The following set

$$Q := \{q \in \mathbb{R} : T(q) = \tilde{t}(q)\}$$

is key to find the phase transitions of $T(q)$. We use Sarig's thermodynamic formalism for countable Markov shifts to prove the existence of the Gibbs state μ_q for $q\phi - T(q)\psi$ for each $q \in Q^{\complement}$ (see Theorem 2.3.29) and recall that ϕ has a Gibbs state μ. These Gibbs states are necessary for the proofs of our main results, Theorems 3.1.5 and 3.1.6.

To form an expression for the multifractal spectrum, we will consider metrics d that satisfy the following. Iommi uses Inequality (3.1.1) to choose the metric for his results (see Definition 2.10 of [Iom05]).

Definition 3.1.3. *Let $\psi : \Sigma \to \mathbb{R}^+$ be locally Hölder and take an arbitrary cylinder $[x_1, ..., x_m] \subset \Sigma$. Define the diameter of $[x_1, ..., x_m]$ as*

$$|[x_1, ..., x_m]| := \sup_{x,y \in [x_1,...,x_m]} d(x,y)$$

for a metric d. We consider metrics such that

$$\frac{1}{C} \prod_{j=0}^{m-1} (\exp(\psi(\sigma^j(y))))^{-1} \le |[x_1, x_2, ..., x_m]| \le C \prod_{j=0}^{m-1} (\exp(\psi(\sigma^j(y))))^{-1}, \tag{3.1.1}$$

for every $y = (x_1, x_2, ..., x_m, y_{m+1}, ...) \in [x_1, ..., x_m]$ and a constant $C > 0$. We will call our function ψ a metric potential.

Given our locally Hölder potential ψ, we choose a general metric d that satisfies Inequality (3.1.1). To explain the significance of Definition 3.1.3, we first give an example of a locally Hölder potential $\tilde{\psi}$ and a metric d on Σ satisfying Inequality (3.1.1). Consider an arbitrary $x = (x_1, x_2, ...) \in \Sigma$. For each $i \in \mathbb{N}$, take $r_i > 0$ such that $\sum_{i=1}^{\infty} r_i = 1$. In many of our examples (see Section 3.11), we will consider a locally constant potential $\tilde{\psi} : \Sigma \to \mathbb{R}^+$:

$$\tilde{\psi}(x) = \log r_{x_1}^{-1} = -\log r_{x_1}. \tag{3.1.2}$$

This potential can be used to approximate ψ by Proposition 2.3.19.

Consider the following metric d on the countable shift Σ. Take the arbitrary sequences $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...) \in \Sigma$. Find the first common, starting subword in which x and y agree. In particular, assume that $x \wedge y = (x_1, ..., x_k)$ such that $k = \max\{m \in \mathbb{N} : x_i = y_i \text{ for all } 1 \le i \le m\}$. Then, we define our metric as

$$d(x,y) = r_{x_1} \cdots r_{x_k} \tag{3.1.3}$$

for $x, y \in \Sigma$.

By Equation (3.1.2),

$$0 \le \prod_{j=0}^{k-1} (\exp(\tilde{\psi}(\sigma^j(x))))^{-1} = (r_{x_1}^{-1} r_{x_2}^{-1} \cdots r_{x_k}^{-1})^{-1} = r_{x_1} r_{x_2} \cdots r_{x_k}. \tag{3.1.4}$$

Given our metric, defined by Equation (3.1.3),

$$|[x_1,...,x_k]| = \sup_{x,y\in[x_1,...,x_k]} d(x,y) = r_{x_1} r_{x_2} \cdots r_{x_k}. \tag{3.1.5}$$

Therefore, by Equations (3.1.4) and (3.1.5),

$$\prod_{j=0}^{k-1} (\exp(\tilde{\psi}(\sigma^j(x))))^{-1} = |[x_1,x_2,...,x_k]|.$$

The equation above relates the potential $\tilde{\psi}$ to the diameter of the cylinder set $[x_1,...,x_m]$. Thus, $\tilde{\psi}$ is a metric potential with respect to the chosen metric d on Σ according to Inequality (3.1.1).

Our choice of metric d allows us to define the multifractal spectrum by using symbolic dimension (see Definition 3.1.4) rather than local dimension. We remark that ψ is often defined by an expanding Markov map. For instance, take an expanding, countably branched Markov map $T:(a,b)\to(a,b)$ for $b>a\geq 0$ modelled by a countable Markov shift Σ. Assume that $\psi = \log|T'\circ\pi|$ such that the coding map $\pi:\Sigma\to(a,b)$. Then, using Inequality (3.1.1), we would choose a metric d such that there exists a constant $C>0$ satisfying

$$-C\exp\left(\sum_{j=0}^{m-1} -\log|T'(T^i(\pi(x)))|\right) \leq |[x_1,x_2,...,x_m]| \leq C\exp\left(\sum_{j=0}^{m-1} -\log|T'(T^i(\pi(x)))|\right)$$

for each $x=(x_1,x_2,...,x_m,...)\in[x_1,x_2,...,x_m]\subset\Sigma$. We explain how our results on the multifractal spectrum's phase transitions (Theorems 3.1.5 and 3.1.6) would be applied to expanding, countably-branched Markov maps after Definition 3.9.3.

We define a notion of dimension by using cylinders.

Definition 3.1.4. *The* symbolic dimension *of a sequence* $x=(x_1,x_2,...,x_m,...)\in\Sigma$ *is*

$$d_\mu(x) := \lim_{m\to\infty} \frac{\log\mu([x_1,x_2,...,x_m])}{\log|[x_1,x_2,...,x_m]|}.$$

We now provide an alternate definition for the multifractal spectrum (see Definition 2.5.8 for the typical definition). Recall the Gibbs measure μ for the potential $\phi:\Sigma\to\mathbb{R}^-$. Consider the values

$$\alpha_{\inf} = \inf\left\{\lim_{m\to\infty}\frac{\log\mu([x_1,x_2,...,x_m])}{\log|[x_1,x_2,...,x_m]|} : [x_1,x_2,...,x_m]\subset\Sigma\right\} \text{ and}$$

$$\alpha_{\sup} = \sup\left\{\lim_{m\to\infty}\frac{\log\mu([x_1,x_2,...,x_m])}{\log|[x_1,x_2,...,x_m]|} : [x_1,x_2,...,x_m]\subset\Sigma\right\}.$$

Consider

$$X_\alpha^s := \left\{x\in\Sigma : \lim_{m\to\infty}\frac{\log\mu([x_1,x_2,...,x_m])}{\log|[x_1,x_2,...,x_m]|} = \alpha\right\}$$

for each $\alpha\in(\alpha_{\inf},\alpha_{\sup})$. Lemma 3.5.5 proves that $X_\alpha^s=\emptyset$ if $\alpha\notin[\alpha_{\inf},\alpha_{\sup}]$. Denote the set

$$X' := \left\{x\in\Sigma : \lim_{m\to\infty}\frac{\log\mu([x_1,x_2,...,x_m])}{\log|[x_1,x_2,...,x_m]|} \text{ does not exist}\right\}.$$

Hence, the shift space

$$\Sigma = \biguplus_{\alpha\in(\alpha_{\inf},\alpha_{\sup})} X^s_\alpha \biguplus X'.$$

Because the symbolic and local dimension are equal on every set with the exception of a set of small Hausdorff dimension (see Proposition 3.5.4), the multifractal spectrum (see Proposition 3.5.7) is

$$f_\mu(\alpha) = \dim_H(X^s_\alpha) \tag{3.1.6}$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$. Theorem 4.1 of [Iom05] states that

$$f_\mu(\alpha) = \inf_{q\in\mathbb{R}}\{T(q) + q\alpha\} \tag{3.1.7}$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$. Our choice of metric d allows us to use the expressions of the multifractal spectrum given by Equations (3.1.6) and (3.1.7).

3.1.2 Methdology and Results

To prove the following theorem about the multifractal spectrum's non-analytic points, we will use results from Sarig [Sar03], Mauldin and Urbański [MU03], and Iommi [Iom05].

Theorem 3.1.5. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to $-\psi$ and the potentials were chosen so that $0 < \alpha_{\lim} < \infty$. Denote μ as the Gibbs state for ϕ.*

1. *There exist intervals A_i such that $f_\mu(\alpha)$ is analytic on each of their interiors.*
2. *The interval $(\alpha_{\inf}, \alpha_{\sup}) = \cup_{i=1}^{j} A_i$ such that $j \in \{1,2,3,4\}$.*
3. *The multifractal spectrum is concave on $(\alpha_{\inf}, \alpha_{\sup})$, has a maximum at a single point, and has zero to three phase transitions.*

We will also prove that the multifractal spectrum has 0 to 1 phase transition if $\alpha_{\lim} = \infty$.

Theorem 3.1.6. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to $-\psi$ and the potentials were chosen so that $\alpha_{\lim} = \infty$. Denote μ as the Gibbs state for ϕ.*

1. *There exist intervals A_i such that $f_\mu(\alpha)$ is analytic on each of their interiors.*
2. *The interval $(\alpha_{\inf}, \alpha_{\sup}) = \cup_{i=1}^{j} A_i$ such that $j \in \{1,2\}$.*
3. *The multifractal spectrum is concave on $(\alpha_{\inf}, \alpha_{\sup})$ and has zero to one phase transition.*

We take the following steps to prove and apply Theorems 3.1.5 and 3.1.6. Section 3.2 proves that local Hölder continuity for our metric d (satisfying Inequality (3.1.1)) implies local Hölder continuity on Sarig's metric (see Equation (3.2.2)). In Section 3.3, we prove results about our potentials ϕ and ψ. These results include the existence of a Gibbs measure μ_q for $q\phi - T(q)\psi$ (see Theorem 2.3.29), an expression for $\tilde{t}(q)$, and conditions for the analyticity of pressure. In that section, the set $Q^{\complement}$ will be key for finding those conditions. Then, Section 3.4 forms results on the analyticity of $T(q)$ and $\alpha(q)$ (which is defined in terms of $-T'(q)$). Section 3.5 proves that our definition of the multifractal spectrum is equivalent to its typical definition. Section 3.6 finds the multifractal spectrum's regions of analyticity by using both $T(q)$ and $\alpha(q)$ and then, proves Theorems 3.1.5 and 3.1.6 on the multifractal spectrum's phase transitions.

Finally, we apply Theorems 3.1.5 and 3.1.6 to the Gauss map G. The continued fraction map $\pi : \mathbb{N}^{\mathbb{N}} \to [0,1]$ is the coding map. Using locally constant potentials on $\mathbb{N}^{\mathbb{N}}$, we provide examples that apply Theorems 3.1.5 and 3.1.6 to illustrate the number of possible phase transitions for the multifractal spectrum when $0 < \alpha_{\lim} \leq \infty$. Also, we provide an example when $\alpha_{\lim}$ does not exist. In this case, the multifractal spectrum can have infinitely many phase transitions. Before we provide more detail on our problem's setting, we discuss a technical point about local Hölder continuity for our chosen general metric d (satisfying Inequality (3.1.1)).

3.2 Local Hölder Continuity and Our Metric

Given our locally Hölder potential ψ, we choose a general metric d that satisfies Inequality (3.1.1). Throughout this chapter, we will use Sarig's thermodynamic formalism [Sar99]. However, we use a different metric d (satisfying Inequality (3.1.1)), which differs from the metric $\tilde{d}$ Sarig (see Equations (3.2.1) and (3.2.2)) uses to develop his results.

Now, we state Sarig's metric. Take a cylinder $[x_1, ..., x_m] \subset \Sigma$ and consider $x, y \in [x_1, ..., x_m] \subset \Sigma$. Define

$$n(x,y) := \inf\{k \in \mathbb{N} : x_k \neq y_k\}. \tag{3.2.1}$$

For our chosen x and y, $n(x,y) = m+1$. The typical metric for the shift is

$$\tilde{d}(x,y) = \left(\frac{1}{2}\right)^{n(x,y)}. \tag{3.2.2}$$

To use Sarig's thermodynamic formalism, we need to show that locally Hölder potentials with respect to our metric d (satisfying Inequality (3.1.1)) are also locally Hölder with respect to $\tilde{d}$. However, this is best explained through an example. Recall the potential $\tilde{\psi}$ given by Equation (3.1.2). Again, take a cylinder $[x_1, ..., x_m] \subset \Sigma$ and consider $x, y \in [x_1, ..., x_m] \subset \Sigma$. For instance, if we take the metric d as given by Equation (3.1.3), it can be shown that there exists a $s \in \mathbb{N}$ such that

$$d(x,y) = (r_{x_1} \cdots r_{x_m})^s \leq \left(\frac{1}{2}\right)^{m+1} \tag{3.2.3}$$

for all $m \in \mathbb{N}$. Because $s\tilde{\psi}$ is locally Hölder (see Equation (3.1.2)) with respect to d, Inequality (3.2.3) gives us that $s\tilde{\psi}$ is locally Hölder with respect to Sarig's metric $\tilde{d}$. Hence, $\tilde{\psi}$ is locally Hölder with respect to both metrics.

We now give a general argument to prove that local Hölder continuity on our metric yields local Hölder continuity on Sarig's metric. To prove that locally Hölder potentials with respect to our metric d (satisfying Inequality (3.1.1)) are also locally Hölder with respect to $\tilde{d}$, we would take our chosen metric potential ψ and general metric d (satisfying Inequality (3.1.1)). Again, take a cylinder $[x_1,....,x_m] \subset \Sigma$ and consider $x, y \in [x_1,....,x_m] \subset \Sigma$. Then, we would find that there exists a $s \in \mathbb{N}$ such that

$$(d(x,y))^s \leq \left(\frac{1}{2}\right)^{m+1} \tag{3.2.4}$$

for all $m \in \mathbb{N}$. Then, $s\psi$ is locally Hölder (see Inequality (3.1.1)) with respect to d, Inequality (3.2.4) gives us that $s\psi$ is locally Hölder with respect to Sarig's metric $\tilde{d}$. Therefore, ψ is locally Hölder with respect to both metrics.

Therefore, by Definition 2.3.17, locally Hölder potentials with respect to our metric d in Equation 3.1.3 are also locally Hölder with respect to the metric $\tilde{d}$. Therefore, Sarig's results on thermodynamic formalism are applicable to our potentials, which are locally Hölder with respect to our metric d. Note that there exist locally Hölder potentials with respect to the typical metric $\tilde{d}$ that are not locally Hölder potentials with respect to our metric d.

Finally, we are ready to introduce our locally Hölder potentials. From these functions, we will get the necessary Gibbs states to prove Theorems 3.1.5 and 3.1.6.

3.3 The Potentials ϕ and ψ

Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$. Later, we will prove that there exists a Gibbs measure μ for ϕ (see Theorem 2.3.29). We will use this measure to form our results about the multifractal spectrum. Also, let $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential (see Definition 3.1.3) such that $\mathscr{P}(-\psi) < \infty$. We assume that ϕ is non-cohomologous (see Definition 2.3.20) to ψ.

3.3.1 Thermodynamic Properties of the Potentials ϕ and ψ

Now, we consider the potential $q\phi - t\psi$ for each fixed $q, t \in \mathbb{R}$. Using this family of potentials, we recall the definition of the function $T(q)$. We need this function to prove Theorems 3.1.5 and 3.1.6 about the multifractal spectrum's phase transitions. Definition 4.2 of [Iom05] gives us the following definition.

Definition 3.3.1. *For each $q \in \mathbb{R}$, the* temperature function

$$T(q) := \inf\{t \in \mathbb{R} : \mathscr{P}(q\phi - t\psi) \leq 0\}$$

The following function $\tilde{t}(q)$ is similarly defined.

Definition 3.3.2. *For each* $q \in \mathbb{R}$, *the* function

$$\tilde{t}(q) := \inf\{t \in \mathbb{R} : \mathscr{P}(q\phi - t\psi) < \infty\}.$$

The set $Q := \{q \in \mathbb{R} : T(q) = \tilde{t}(q)\}$ will help us find the multifractal spectrum's non-analytic points. We used pressure to define the function $\tilde{t}(q)$. Hence, before forming an expression for $\tilde{t}(q)$, we give a necessary definition and lemma on pressure.

Definition 3.3.3. *Consider a potential* $f : \Sigma \to \mathbb{R}$. *The* nth partition function *is*

$$Z_m(f) = \sum_{[x_1,\ldots,x_m]\subset\Sigma} \exp \sup_{y\in[x_1,\ldots,x_m]} \left(\sum_{i=0}^{m-1} f\left(\sigma^i(y)\right) \right).$$

The following lemma is a modified version of Proposition 2.1.9 on Page 11 in Mauldin and Urbański [MU03].

Lemma 3.3.4. *If* Σ *is topologically mixing and satisfies the BIP property and* $f : \Sigma \to \mathbb{R}$ *is locally Hölder, then* $\mathscr{P}(f) < \infty$ *if and only if* $Z_1(f) < \infty$.

By Lemma 3.3.4, we find that

$$\tilde{t}(q) = \inf\{t \in \mathbb{R} : \mathscr{P}(q\phi - t\psi) < \infty\} = \inf\{t \in \mathbb{R} : Z_1(q\phi - t\psi) < \infty\}. \tag{3.3.1}$$

The following limit is used to form an equation for $\tilde{t}(q)$.

Definition 3.3.5. *Take an arbitrary sequence* $\hat{i} = (i, y_2, y_3, \ldots) \in \Sigma$ *for each* $i \in \mathbb{N}$. *Let*

$$\alpha_{\lim} := \lim_{i\to\infty} \frac{\phi(\hat{i})}{-\psi(\hat{i})}$$

if the limit exists.

First, we justify choosing an arbitrary $\hat{i} = (i, y_2, y_3, \ldots) \in \Sigma$ to define $\alpha_{\lim}$. Then, we will give conditions that make it possible for $0 < \alpha_{\lim} \le \infty$. Because ϕ and ψ are locally Hölder,

$$|\phi(\hat{i}) - \phi(j)| \le V_1(\phi) < \infty \text{ and } |-\psi(\hat{i}) + \psi(j)| \le V_1(-\psi) < \infty \tag{3.3.2}$$

for any $j \in [i]$. Hence, the limit $\alpha_{\lim}$ is independent of the choice of $j \in [i]$ because ϕ and ψ are locally Hölder.

Now, we prove that our assumption, $\mathscr{P}(-\psi) < \infty$, ensures that $\alpha_{\lim} \in (0, \infty]$ is possible. We need to assume that $\mathscr{P}(\phi) = 0$ because this will ensure that ϕ has a Gibbs state (see Theorem 3.3.7). Take an arbitrary $\hat{i} = (i, y_2, y_3, \ldots) \in \Sigma$ for each $i \in \mathbb{N}$. Because $\mathscr{P}(\phi) = 0$,

$$Z_1(\phi) = \sum_{x_1=1}^{\infty} \exp\left(\sup_{x\in[x_1]} \phi(x) \right) < \infty. \tag{3.3.3}$$

by Lemma 3.3.12. We find that

$$\lim_{i\to\infty} \phi(\hat{i}) = -\infty \tag{3.3.4}$$

because of Equation (3.3.3) and Inequality (3.3.2). Because of Equation (3.3.4) and Inequality (3.3.2), we need to assume that

$$\lim_{i\to\infty} -\psi(\hat{i}) = -\infty, \tag{3.3.5}$$

so that it is possible for $0 < \alpha_{\lim} \leq \infty$. Then, Equation (3.3.5) is true if and only if

$$Z_1(-\psi) = \sum_{x_1=1}^{\infty} \exp\left(\sup_{x\in[x_1]} -\psi(x)\right) < \infty. \tag{3.3.6}$$

Because of Lemma 3.3.4, Equation (3.3.6) is equivalent to $\mathscr{P}(-\psi) < \infty$. Thus, we must assume our potential ψ satisfies $\mathscr{P}(-\psi) < \infty$.

Consider

$$t_\infty := \inf\{t \in \mathbb{R} : \mathscr{P}(-t\psi) < \infty\}.$$

We find that $t_\infty \leq 1$ because $\mathscr{P}(-\psi) < \infty$. By Lemma 3.3.4, we find that

$$t_\infty = \inf\{t \in \mathbb{R} : \mathscr{P}(-t\psi) < \infty\} = \inf\{t \in \mathbb{R} : Z_1(-t\psi) < \infty\}. \tag{3.3.7}$$

The value t_∞ is used to form an equation for $\tilde{t}(q)$.

Proposition 3.3.6. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to ψ and the potentials were chosen so that $0 < \alpha_{\lim} < \infty$. Then,*

$$\tilde{t}(q) = -\alpha_{\lim} q + t_\infty.$$

Furthermore, $\tilde{t}(q)$ is a decreasing line.

Proof. We give an outline of our proof. By Proposition 2.3.19, we can respectively approximate our locally Hölder potentials ϕ and ψ with locally constant potentials $\tilde{\phi}$ and $\tilde{\psi}$. Then, we re-express $\tilde{t}(q)$, $\alpha_{\lim}$, and t_∞ by using these locally constant potentials. We prove that $\tilde{t}(q) \leq t_\infty - q\alpha_{\lim}$ and finally, prove that the inequality is instead an equality. Now, we start the proof of this proposition.

Assume that the locally constant functions $\tilde{\phi}$ and $\tilde{\psi}$ approximate ϕ and ψ, i.e. there exists a sufficiently small $\varepsilon > 0$ such that

$$|\phi(x) - \tilde{\phi}(x)| \leq \varepsilon \text{ and } |\psi(x) - \tilde{\psi}(x)| \leq \varepsilon. \tag{3.3.8}$$

Such an $\varepsilon > 0$ exists because $\phi - \tilde{\phi}$ and $\psi - \tilde{\psi}$ are locally Hölder. Furthermore, assume that these locally constant functions satisfy the following without loss of generality. For each $i \in \mathbb{N}$, take $c_i > 0$ and $d_i < 0$. Let $\tilde{\psi}$ and $\tilde{\phi}$ be locally constant functions such that

$$\tilde{\psi}(x) = c_i \text{ and } \tilde{\phi}(x) = d_i \tag{3.3.9}$$

for every $x \in [i]$.

Fix any $q, t \in \mathbb{R}$. By Inequality (3.3.8), there exists a sufficiently small $\tilde{\varepsilon} = (|q| + |t|)\varepsilon > 0$ such that

$$|q\phi(x) - t\psi(x) - [q\tilde{\phi}(x) - t\tilde{\psi}(x)]| \le \tilde{\varepsilon} \tag{3.3.10}$$

for each $x \in \Sigma$. Hence,

$$\tilde{t}(q) = \inf\{t \in \mathbb{R} : Z_1(q\phi - t\psi) < \infty\} = \inf\{t \in \mathbb{R} : Z_1(q\tilde{\phi} - t\tilde{\psi}) < \infty\} \tag{3.3.11}$$

by Inequality (3.3.10).

Now, we find an expression for $Z_1(q\tilde{\phi} - t\tilde{\psi})$. Take an arbitrary $\hat{i} = (i, y_2, y_3, ...)$ for each $i \in \mathbb{N}$. By construction of $\tilde{\phi}$ and $\tilde{\psi}$ (see Equation (3.3.9)),

$$Z_1(q\tilde{\phi} - t\tilde{\psi}) = \sum_{x_1=1}^{\infty} \exp\left(\sup_{x \in [x_1]} (q\tilde{\phi} - t\tilde{\psi})(x)\right) = \sum_{i=1}^{\infty} \exp((q\tilde{\phi} - t\tilde{\psi})(\hat{i})). \tag{3.3.12}$$

By Equation 3.3.12, we need to find the value

$$\tilde{t}(q) = \inf\left\{t \in \mathbb{R} : \sum_{i=1}^{\infty} \exp((q\tilde{\phi} - t\tilde{\psi})(\hat{i})) < \infty\right\} \tag{3.3.13}$$

for each $q \in \mathbb{R}$. To do this, we need to use the definition of $\alpha_{\lim}$. For each $n \in \mathbb{N}$, take an arbitrary sequence $\hat{n} = (n, y_2, y_3, ...) \in \Sigma$. By assumption, Definition 3.3.5, and Equation (3.3.9), for each $n \ge N$, there exists an $\varepsilon(n) > 0$ such that

$$\frac{\tilde{\phi}(\hat{n})}{-\tilde{\psi}(\hat{n})} - \varepsilon(n) \le \alpha_{\lim} \le \frac{\tilde{\phi}(\hat{n})}{-\tilde{\psi}(\hat{n})} + \varepsilon(n). \tag{3.3.14}$$

Hence,

$$\tilde{t}(q) = \inf\left\{t \in \mathbb{R} : \sum_{i=1}^{\infty} \exp((q\tilde{\phi} - t\tilde{\psi})(\hat{i})) < \infty\right\} = \inf\left\{t \in \mathbb{R} : \sum_{n=N}^{\infty} \exp((q\tilde{\phi} - t\tilde{\psi})(\hat{n})) < \infty\right\}. \tag{3.3.15}$$

Now, we re-express t_∞ by using the locally constant potential $\tilde{\psi}$ because that value will be used in our expression for $\tilde{t}(q)$. Recall that Equation 3.3.7 states that

$$t_\infty = \inf\{t \in \mathbb{R} : \mathscr{P}(-t\psi) < \infty\} = \inf\{t \in \mathbb{R} : Z_1(-t\psi) < \infty\}. \tag{3.3.16}$$

Because of Equations (3.3.16) and (3.3.9) and our choice of sufficiently large $n \in \mathbb{N}$,

$$t_\infty = \inf\{t \in \mathbb{R} : Z_1(-t\tilde{\psi}) < \infty\} = \inf\left\{t \in \mathbb{R} : \sum_{n=N}^{\infty} \exp((-t\tilde{\psi})(\hat{n})) < \infty\right\}. \tag{3.3.17}$$

Without loss of generality, fix any $q \ge 0$. First, we prove that $\tilde{t}(q) \le t_\infty - q\alpha_{\lim}$. If $t = t_\infty - q\alpha_{\lim}$,

$$\begin{aligned}
\sum_{n=N}^{\infty} \exp((q\tilde{\phi} - t\tilde{\psi})(\hat{n})) &= \sum_{n=N}^{\infty} \exp((q\tilde{\phi} + (-t_\infty + q\alpha_{\lim})\tilde{\psi})(\hat{n})) \\
&\le \sum_{n=N}^{\infty} \exp\left(\left(q\tilde{\phi} + \left(-t_\infty + q\left(\frac{\tilde{\phi}(\hat{n})}{-\tilde{\psi}(\hat{n})} + \varepsilon(n)\right)\tilde{\psi}\right)\right)(\hat{n})\right) \\
&= \sum_{n=N}^{\infty} \exp\left((-t_\infty + q\varepsilon(n))\tilde{\psi}(\hat{n})\right) \\
&= \sum_{n=N}^{\infty} \exp\left((-(t_\infty - q\varepsilon(n)))\tilde{\psi}(\hat{n})\right) = \infty \qquad (3.3.18)
\end{aligned}$$

by Inequalities (3.3.14) and (3.3.17). Hence,

$$\tilde{t}(q) \leq t_\infty - q\alpha_{\text{lim}}. \tag{3.3.19}$$

Finally, we prove that Inequality (3.3.19) is actually an equality. Now, consider $t = t_\infty - q\alpha_{\text{lim}} + K$ such that $K \in \mathbb{R}$. We find that

$$\lim_{n\to\infty} \varepsilon(n) = 0 \tag{3.3.20}$$

by Inequality (3.3.14). By Equation (3.3.20) and Inequality (3.3.18), there exists a sufficiently small $\delta(N) > 0$ such that

$$\left| \sum_{n=N}^{\infty} \exp((q\tilde{\phi} - t\tilde{\psi})(\hat{n})) - \sum_{n=N}^{\infty} \exp(-(t_\infty + K)\tilde{\psi}(\hat{n})) \right| \leq \delta(N). \tag{3.3.21}$$

Furthermore, we find that

$$\sum_{n=N}^{\infty} \exp(-(t_\infty + K)\tilde{\psi}(\hat{n})) < \infty \tag{3.3.22}$$

if $K > 0$ and

$$\sum_{n=N}^{\infty} \exp(-(t_\infty + K)\tilde{\psi}(\hat{n})) = \infty \tag{3.3.23}$$

if $K < 0$ by Inequality (3.3.17).

Therefore,

$$\tilde{t}(q) = \inf\{t \in \mathbb{R} : Z_1(q\phi - t\psi) < \infty\} = \inf\left\{ t \in \mathbb{R} : \sum_{n=N}^{\infty} \exp((q\tilde{\phi} - t\tilde{\psi})(\hat{n})) < \infty \right\} = t_\infty - q\alpha_{\text{lim}}. \tag{3.3.24}$$

by Equation (3.3.15), Inequalities (3.3.18) and (3.3.22), and Equation (3.3.23).

Furthermore, $\tilde{t}(q) = -\alpha_{\text{lim}} q + t_\infty$ is a decreasing line because $\tilde{t}'(q) = -\alpha_{\text{lim}} < 0$. ■

The equation for $\tilde{t}(q)$, given by Proposition 3.3.6, will help us prove that it is possible for $T(q)$ to have phase transitions (see Proposition 3.4.3). In turn, this will help us prove Theorem 3.1.5 on the analytic regions and phase transitions of the multifractal spectrum. We will frequently use the set

$$Q^{\complement} = \{q \in \mathbb{R} : T(q) \neq \tilde{t}(q)\}$$

to do this. Now, we need to establish the existence of Gibbs measures for our potentials and find regions where pressure is analytic.

3.3.2 Gibbs Measures and Analyticity of Pressure

First, we prove that there exists a Gibbs measure μ for ϕ. Fix $q \in Q^{\complement}$. Then, we prove that there exists a Gibbs measure μ_q for $q\phi - T(q)\psi$. Next, we prove $t \mapsto \mathscr{P}(q\phi - T(q)\psi)$ is analytic in a neighbourhood of $T(q)$. Next, we analyse the analyticity of $q \mapsto \mathscr{P}(q\phi - t\psi)$ in a neighbourhood of q. Finally, we use the analyticity of pressure in these neighbourhoods to prove that $\phi, -\psi \in \mathscr{L}^1(\mu_q)$.

By adapting Theorem 2.3.30 to our setting, we find that there exists a Gibbs state μ for ϕ.

Theorem 3.3.7. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be locally Hölder such that $\mathscr{P}(\phi) = 0$. Then, there exists a unique invariant, ergodic Gibbs state μ for ϕ. If this measure satisfies $\int \phi \, \mathrm{d}\mu > -\infty$, μ is the unique equilibrium state for ϕ.*

Fix $q \in Q^{\complement}$. Before we prove that there exists a Gibbs measure for $q\phi - T(q)\psi$ (see Theorem 3.3.10), we will form a more precise expression for $Q^{\complement}$ and prove that $\mathscr{P}(q\phi - t\psi) < \infty$ on $(\tilde{t}(q), \infty)$. Fix $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), T(s))$. We will also prove that $\mathscr{P}(q\phi - t\psi) < \infty$ on (s, ∞) because this will help us prove that this $q \mapsto \mathscr{P}(q\phi - t\psi)$ is analytic on a neighbourhood in (s, ∞).

Lemma 3.3.8. *Fix $q \in Q^{\complement}$. For each $t \in (\tilde{t}(q), \infty)$,*

$$\mathscr{P}(q\phi - t\psi) < \mathscr{P}(q\phi - \tilde{t}(q)\psi) < \infty.$$

Furthermore,

$$t \mapsto \mathscr{P}(q\phi - t\psi)$$

is a decreasing convex function on $(\tilde{t}(q), \infty)$.

Proof. Fix $q \in Q^{\complement}$. Take $\tilde{t}(q) < t_1 < t_2 < \infty$. Because $\psi : \Sigma \to \mathbb{R}^+$, we immediately find that

$$\mathscr{P}(q\phi - t_2\psi) < \mathscr{P}(q\phi - t_1\psi) < \mathscr{P}(q\phi - \tilde{t}(q)\psi) < \infty \tag{3.3.25}$$

by the definition of pressure (see Definition 2.3.21) and construction of $\tilde{t}(q)$. Furthermore,

$$t \mapsto \mathscr{P}(q\phi - t\psi)$$

is a decreasing function on $(\tilde{t}(q), \infty)$ by Inequality (3.3.25). ■

Fix $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), T(s))$. Now, we prove that

$$q \mapsto \mathscr{P}(q\phi - t\psi)$$

is a decreasing function on (s, ∞)

Lemma 3.3.9. *Fix $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), T(s))$. For each $q \in (s, \infty)$,*

$$\mathscr{P}(q\phi - t\psi) < \mathscr{P}(s\phi - \tilde{t}(s)\psi) < \infty.$$

Furthermore,

$$q \mapsto \mathscr{P}(q\phi - t\psi)$$

is a decreasing convex function on (s, ∞).

Proof. Fix $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), T(s))$. Take $s < q_1 < q_2 < \infty$. Because $\psi : \Sigma \to \mathbb{R}^-$, we immediately find that

$$\mathscr{P}(q_2\phi - t\psi) < \mathscr{P}(q_1\phi - t\psi) < \mathscr{P}(s\phi - t\psi) \tag{3.3.26}$$

by the definition of pressure (see Definition 2.3.21). By Lemma 3.3.8 and construction of $\tilde{t}(q)$, we find that

$$\mathscr{P}(s\phi - t\psi) < \mathscr{P}(s\phi - \tilde{t}(s)\psi) < \infty.$$

Furthermore,

$$q \mapsto \mathscr{P}(q\phi - t\psi)$$

is a decreasing convex function on (s, ∞) by Inequality (3.3.26). ■

Because of these preceding results on the behaviour of pressure, we recall the definition of $Q^{\complement}$. The set

$$Q^{\complement} = \{q \in \mathbb{R} : T(q) \neq \tilde{t}(q)\} = \{q \in \mathbb{R} : T(q) > \tilde{t}(q)\}$$

by Lemma 3.3.8. We need to analyse the family of potentials $\left\{q\phi - T(q)\psi : q \in Q^{\complement}\right\}$ because results for these potentials will give us information about the possible analytic regions of the multifractal spectrum. Now, we prove the existence and uniqueness of Gibbs measures for each potential in this family.

Theorem 3.3.10. *For each $q \in Q^{\complement}$, there exists a unique invariant, ergodic Gibbs state μ_q for $q\phi - T(q)\psi$. If this measure satisfies $\int q\phi - T(q)\psi \, \mathrm{d}\mu_q > -\infty$, μ_q is the unique equilibrium state for $q\phi - T(q)\psi$.*

Proof. Fix any $q \in Q^{\complement}$. Because (Σ, σ) is topologically mixing and satisfies the BIP property, $q\phi - T(q)\psi$ is locally Hölder, and $\mathscr{P}(q\phi - T(q)\psi) = 0$, the potential $q\phi - T(q)\psi$ has a unique invariant, ergodic Gibbs state μ_q by Theorem 2.3.30. Furthermore, if $-q\phi + T(q)\psi$ is integrable, the measure μ_q is the unique equilibrium state for $q\phi - T(q)\psi$ by the same theorem. ■

We will later prove that $\mu_q(X^s_{\alpha(q)}) = 1$ (see Proposition 3.5.4). Before we provide a proposition, resulting from Theorem 3.3.10, on the analyticity of $t \mapsto \mathscr{P}(q\phi - t\psi)$ for each $q \in Q^{\complement}$, we adapt Definition 2.3.31 to our setting and prove that this function is finite on a neighbourhood of $T(q)$.

Definition 3.3.11. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Fix $q \in Q^{\complement}$ and let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential. Denote $Dir(\phi)$ as the collection of all $\psi : \Sigma \to \mathbb{R}^+$ such that there exists $C_\psi > 0, r \in (0,1)$ and sufficiently small $\varepsilon > 0$ satisfying*

1. *$V_m(\psi) < C_\psi r^m$ for all $m \geq 1$*
2. *$\mathscr{P}(q\phi - t\psi) < \infty$ for all $t \in (T(q) - \varepsilon, T(q) + \varepsilon)$.*

Fix $q \in Q^{\complement}$. According to Proposition 2.3.32, we will need to show that $\psi \in Dir(\phi)$ to prove that $t \mapsto \mathscr{P}(q\phi - t\psi)$ is analytic on an ε−neigbourhood of $T(q)$.

Proposition 3.3.12. *For each $q \in Q^{\complement}$, there exists a sufficiently small $\varepsilon > 0$ such that $t \mapsto \mathscr{P}(q\phi - t\psi)$ is real analytic on $(T(q) - \varepsilon, T(q) + \varepsilon)$.*

Proof. Fix $q \in Q^{\complement}$. We find that $\psi \in Dir(\phi)$ because of Lemma 3.3.8 and it is locally Hölder. Therefore, there exists a sufficiently small $\varepsilon > 0$ such that $t \mapsto \mathscr{P}(q\phi - t\psi)$ is real analytic on $(T(q) - \varepsilon_0, T(q) + \varepsilon_0)$ by Proposition 2.3.32. ■

We now prepare to prove a result on the analyticity of $q \mapsto \mathscr{P}(q\phi - t\psi)$. Consider the following definition, which adapts Definition 2.3.31 to our setting.

Definition 3.3.13. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Let $\psi : \Sigma \to \mathbb{R}^+$ be a locally Hölder potential such that $\mathscr{P}(-\psi) < \infty$. Fix $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), \infty)$. Denote $Dir(\psi)$ as the collection of all $\phi : \Sigma \to \mathbb{R}^-$ such that there exists $C_\phi > 0, r \in (0,1)$ and sufficiently small $\varepsilon > 0$ satisfying*

1. *$V_m(\phi) < C_\phi r^m$ for all $m \geq 1$*
2. *$\mathscr{P}(q\phi - t\psi) < \infty$ for all $q \in (s, s + \varepsilon)$.*

Fix $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), \infty)$. According to Proposition 2.3.32, we will need to show that $\phi \in Dir(\psi)$ to prove that there exists an $\varepsilon > 0$ such that $q \mapsto \mathscr{P}(q\phi - t\psi)$ is analytic on $(s, s + \varepsilon)$.

Proposition 3.3.14. *Fix $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), \infty)$. Then, there exists a sufficiently small $\varepsilon > 0$ such that $q \mapsto \mathscr{P}(q\phi - t\psi)$ is real analytic on $(s, s + \varepsilon)$.*

Proof. Fix $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), \infty)$. We find that $\phi \in Dir(\psi)$ because of Lemma 3.3.9 and it is locally Hölder. Therefore, there exists a sufficiently small $\varepsilon > 0$ such that $q \mapsto \mathscr{P}(q\phi - t\psi)$ is real analytic on $(s, s + \varepsilon)$ by Proposition 2.3.32. ■

Given our analyticity results, we prove that $\phi, -\psi \in \mathscr{L}^1(\mu_q)$ if $q \in Q^{\complement}$.

Proposition 3.3.15. *For each $q \in Q^{\complement}$, $\phi, -\psi \in \mathscr{L}^1(\mu_q)$.*

Proof. First, we prove that $-\psi \in \mathscr{L}^1(\mu_q)$. Fix an arbitrary $q \in Q^{\complement}$. There exists a Gibbs measure μ_q for $q\phi - T(q)\psi$ by Theorem 3.3.10. There exists a sufficiently small $\varepsilon > 0$ such that $t \mapsto \mathscr{P}(q\phi - t\psi)$ is analytic on $(T(q) - \varepsilon, T(q) + \varepsilon)$ by Proposition 3.3.12. Because of Theorem 2.3.33 and $t \mapsto \mathscr{P}(q\phi - t\psi)$ is convex,

$$\frac{\partial \mathscr{P}(q\phi - t\psi)}{\partial t} = -\int_\Sigma \psi \, d\mu_q < 0 \tag{3.3.27}$$

for each $t \in (T(q) - \varepsilon, T(q) + \varepsilon)$. Hence, $-\psi \in \mathscr{L}^1(\mu_q)$ by Equation (3.3.27).

Now, we prove that $\phi \in \mathscr{L}^1(\mu_q)$. Fix arbitrary $s \in Q^{\complement}$ and $t \in (\tilde{t}(s), T(s))$. There exists a sufficiently small $\varepsilon > 0$ such that $q \mapsto \mathscr{P}(q\phi - t\psi)$ is analytic on $(s, s+\varepsilon)$. By Theorem 2.3.33,

$$\frac{\partial \mathscr{P}(q\phi - t\psi)}{\partial q} = \int_{\Sigma} \phi \, \mathrm{d}\mu_q < 0 \tag{3.3.28}$$

for each $q \in (s, s+\varepsilon)$. Because $q \mapsto \mathscr{P}(q\phi - t\psi)$ is convex, $\phi \in \mathscr{L}^1(\mu_q)$ by Equation (3.3.28). ■

In the next section, we will prove results on the phase transitions of $T(q)$. Using $-T'(q)$, we will define the function $\alpha(q)$. Then, we will prove that $\alpha(q)$ can be expressed by using integrals of ϕ and $-\psi$. This function is closely related to the pointwise dimension of μ_q- typical $x \in \Sigma$ (see Proposition 3.5.4). Finding the analytic regions of $T(q)$ and $\alpha(q)$ will help us find the phase transitions of the multifractal spectrum.

3.4 Analyticity of $T(q)$ and $\alpha(q)$

Using $T(q)$, we will construct a function $\alpha(q)$ because it will help us find analytic regions of the multifractal spectrum. First, we provide results about the behaviour of $T(q)$ and find that Q can be an interval.

3.4.1 Phase Transitions of $T(q)$

Iommi (see Proposition 4.3 on Page 1892 of [Iom05]) proves the following important result.

Proposition 3.4.1. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential. Assume that ϕ is non-cohomologous to ψ. Then, $T(q)$ is a convex and decreasing function.*

Because $T(q)$ is convex and decreasing by Proposition 3.4.1 and $\tilde{t}(q)$ is linear when $0 < \alpha_{\lim} < \infty$ by Proposition 3.3.6, we immediately find that

$$Q = \{q \in \mathbb{R} : T(q) = \tilde{t}(q)\}$$

can be an interval, point, or the empty set.

Proposition 3.4.2. *Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to ψ and the potentials were chosen such that $0 < \alpha_{\lim} < \infty$. Then, there exist $q_0, q_1 \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $Q = [q_0, q_1]$. Hence, Q is either a closed interval, half-open infinite interval, a point, or the empty set.*

Then, Proposition 3.4.2 gives us an upper bound for the number of phase transitions of the temperature function.

Proposition 3.4.3. *Assume that (Σ,σ) satisfies the BIP property and is topologically mixing. Let $\phi:\Sigma\to\mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi)=0$ and $\psi:\Sigma\to\mathbb{R}^+$ be a metric potential. Assume that ϕ is non-cohomologous to ψ and the potentials were chosen such that $0<\alpha_{\lim}<\infty$. Then, $T(q)$ has at most two phase transitions.*

Take I to be a countable index set, which contains at least two elements. We remark that Hanus, Mauldin, and Urbański [HMU02] considered families of potentials $\{f^{(i)}: i\in I\}$, which are strongly Hölder, and $\log|\phi_i'|$, which is defined by a regular, conformal iterated function system $\{\phi_i\}_{i\in I}$ that satisfies the open set condition (both the terms regular, conformal iterated function system and the open set condition are defined in Chapter 4 of [MU03]). Their potentials give that $Q=\emptyset$. Hence, the multifractal spectrum is analytic in their case (see Proposition 3.7.3 for an analysis of the multifractal spectrum when $Q=\emptyset$).

Without loss of generality, let $Q=[q_0,q_1]$ for some $q_0,q_1\in\mathbb{R}$. Before defining a key function used in standard multifractal analysis, $\alpha(q)$, we need to prove that $T(q)$ is analytic on open subintervals of $Q^{\complement}$ and Q respectively.

Proposition 3.4.4. *The function $T(q)$ is decreasing, well-defined, and analytic on open subintervals of $Q^{\complement}$ and Q respectively.*

Proof. First, we analyse the behaviour of $T(q)$ on $Q^{\complement}$. Fix an arbitrary q in an open sub-interval of $Q^{\complement}$. By construction of $T(q)$,

$$\mathscr{P}(q\phi-T(q)\psi)=0$$

Hence, $T(q)$ is analytic by the implicit function theorem. By Proposition 3.4.1, $T(q)$ is convex, well-defined, and decreasing on open sub-intervals of $Q^{\complement}$.

Now, we analyse the behaviour of $T(q)$ on Q. Proposition 3.3.6 states that

$$\tilde{t}(q)=t_\infty-q\alpha_{\lim}. \tag{3.4.1}$$

Hence, $\tilde{t}(q)$ is a line on $\mathbb{R}$ and $\tilde{t}'(q)=-\alpha_{\lim}<0$. Because $T(q)=\tilde{t}(q)$ on Q, the function $T(q)$ is decreasing, well-defined, and analytic on open sub-intervals of Q. ■

Finally, we will take the function $\alpha(q):=-T'(q)$ for all $q\in\mathbb{R}$ except q_0 and q_1 because they are the phase transitions of $T(q)$ by Propositions 3.4.1 and 3.3.6. We will use this function to find the multifractal spectrum's phase transitions.

3.4.2 The Function $\alpha(q)$ and its Analytic Regions

We will now define define the function $\alpha(q)$, using $T(q)$, and then, give results about its analytic regions and form an expression for it. Assume that $Q=[q_0,q_1]$ for some $q_0,q_1\in\mathbb{R}$.

Definition 3.4.5. *Assume that ϕ is non-cohomologous to ψ. Take the function*

$$\alpha(q) := \begin{cases} -T'(q) & \text{if } q \in Q^{\complement} \\ \alpha_{\lim} & \text{if } q \in (q_0, q_1) \\ \lim_{q \to q_0^-} -T'(q) & \text{if } q = q_0 > -\infty \\ \lim_{q \to q_1^+} -T'(q) & \text{if } q = q_1 < \infty. \end{cases}$$

Let $\alpha^- := \lim_{q \to q_0^-} -T'(q)$ and $\alpha^+ := \lim_{q \to q_1^+} -T'(q)$. If $q_0 = -\infty$, take $\alpha(q_0) := \alpha_{\lim}$ and if $q_1 = \infty$, take $\alpha(q_1) := \alpha_{\lim}$. Furthermore, if $q_0 = -\infty$ or $q_1 = \infty$, $\alpha(q)$ is defined as above for all other values of q.

We defined $\alpha(q)$ as $\alpha(q_0) := \lim_{q \to q_0^-} -T'(q)$ on q_0 because $T(q)$ is convex, decreasing, and differentiable (see Proposition 3.3.6 and 3.4.4) from the left. The reasoning for the definition of $\alpha(q_1)$ is similar. If Q is a singleton, then $q_0 = q_1 < \infty$. In that case, $\alpha^- = \alpha^+ = \alpha_{\lim}$. Our analysis of $\alpha(q)$ also depends on its extreme values. Let

$$\alpha_{\inf} := \inf_{q \in \mathbb{R}} \alpha(q) \text{ and } \alpha_{\sup} := \sup_{q \in \mathbb{R}} \alpha(q).$$

First, we prove that $\alpha(q)$ is analytic on open sub-intervals of $Q^{\complement}$.

Proposition 3.4.6. *The function $\alpha(q)$ is decreasing, well-defined, and analytic on open sub-intervals of $Q^{\complement}$. Furthermore, $\alpha(q)$ is analytic and constant on open sub-intervals of Q.*

Proof. By Propositions 3.4.4 and 3.4.1, $T(q)$ is decreasing, well-defined, convex, and analytic on open sub-intervals of $Q^{\complement}$. Then, $-T''(q) < 0$. Therefore, $\alpha(q)$ is well-defined, decreasing, and analytic on open sub-intervals of $Q^{\complement}$. By definition, $\alpha(q)$ is constant and analytic on the interior of Q. ■

We form an expression for $\alpha(q)$ in terms of ϕ and ψ.

Proposition 3.4.7. *Fix $q \in Q^{\complement}$. Denote μ_q as the Gibbs state for $q\phi - T(q)\psi$. Then,*

$$\alpha(q) = \frac{\int \phi \, \mathrm{d}\mu_q}{-\int \psi \, \mathrm{d}\mu_q} = -T'(q).$$

Proof. Fix $q \in Q^{\complement}$. There exists a Gibbs measure μ_q for $q\phi - T(q)\psi$ by Theorem 3.3.10. Equation (3.3.27) states that

$$\frac{\partial \mathscr{P}(q\phi - t\psi)}{\partial t}\Big|_{t=T(q)} = -\int_{\Sigma} \psi \, \mathrm{d}\mu_q < 0 \tag{3.4.2}$$

and Equation (3.3.28) states that

$$\frac{\partial \mathscr{P}(s\phi - t\psi)}{\partial s}\Big|_{s=q} = \int_{\Sigma} \phi \, \mathrm{d}\mu_q < 0. \tag{3.4.3}$$

Therefore,

$$\alpha(q) = -T'(q) = \frac{\mathrm{d}t}{\mathrm{d}q}\Big|_{t=T(q)} = \frac{\int \phi \mathrm{d}\mu_q}{-\int \psi \mathrm{d}\mu_q}.$$

■

We prove the following formulae for $\alpha_{\inf}$ and $\alpha_{\sup}$.

Lemma 3.4.8. *We find that*

$$\alpha_{\sup} = \sup_{\nu \in M_\sigma(\Sigma)} \frac{\int \phi \, d\nu}{-\int \psi \, d\nu} \text{ and } \alpha_{\inf} = \inf_{\nu \in M_\sigma(\Sigma)} \frac{\int \phi \, d\nu}{-\int \psi \, d\nu}.$$

Proof. By Proposition 3.4.7,

$$\sup_{\nu \in M_\sigma(\Sigma)} \frac{\int \phi \, d\nu}{-\int \psi \, d\nu} \geq -T'(q) \tag{3.4.4}$$

for all $q \in Q^{\complement}$.

Because we assumed that $q = [q_0, q_1] \subset (-\infty, \infty)$, Propositions 3.4.1 and 3.4.3 give us that

$$\alpha_{\inf} \leq \alpha_{\lim} \leq \alpha_{\sup}. \tag{3.4.5}$$

Therefore,

$$\sup_{\nu \in M_\sigma(\Sigma)} \frac{\int \phi \, d\nu}{-\int \psi \, d\nu} = \sup_{q \in \mathbb{R}} \alpha(q) = \alpha_{\sup}$$

by Inequalities (3.4.4) and (3.4.5) and Definition 3.4.5.

Using a similar argument, the result for the infimum of $\alpha(q)$ also follows:

$$\inf_{\nu \in M(\Sigma, \sigma)} \frac{\int \phi \, d\nu}{-\int \psi \, d\nu} = \inf_{q \in \mathbb{R}} \alpha(q) = \alpha_{\inf}.$$

■

In general, $Q = [q_0, q_1] \cup \{-\infty, \infty\}$ for some $q_0, q_1 \in \mathbb{R}$ by Proposition 3.4.2. Hence, by that proposition, Definition 3.4.5, and Lemma 3.4.8,

$$(\alpha_{\inf}, \alpha_{\sup}) = \{\alpha(q) : q \in Q^{\complement}\} \cup (\alpha^+, \alpha_{\lim}) \cup \{\alpha_{\lim}\} \cup (\alpha_{\lim}, \alpha^-) \cup \{\alpha^-\} \cup \{\alpha^+\}. \tag{3.4.6}$$

We provide Equation (3.4.6) because we will later use it to find the analytic regions of the multifractal spectrum. We find that $\alpha(q)$ is non-negative for every $q \in \mathbb{R} \cup \{-\infty, \infty\}$ because $T(q)$ is a decreasing function of q (see Proposition 3.4.1). We will later find that the expression for the multifractal spectrum uses $T(q)$ and $\alpha(q)$ (see Theorem 3.5.10).

3.5 The Multifractal Spectrum

Before defining the multifractal spectrum, we define symbolic dimension and the set X_α^s. Denote μ as the Gibbs measure for ϕ.

Definition 3.5.1. *The* symbolic dimension *of a sequence* $x = (x_1, x_2, \ldots, x_m, \ldots) \in \Sigma$ *is*

$$d_\mu(x) := \lim_{m \to \infty} \frac{\log \mu([x_1, x_2, \ldots, x_m])}{\log |[x_1, x_2, \ldots, x_m]|}.$$

Similarly, we consider another type of dimension for each $x \in \Sigma$.

Definition 3.5.2. *The* local dimension *of a sequence* $x = (x_1, x_2, \ldots, x_m, \ldots) \in \Sigma$ *is*

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

We now consider sets with symbolic and local dimension α as follows.

Definition 3.5.3. *For each fixed* $\alpha \in [\alpha_{\inf}, \alpha_{\sup}]$, *define the sets*

$$X_\alpha^s := \left\{ x = (x_1, \ldots, x_m, \ldots) \in \Sigma : \lim_{m \to \infty} \frac{\log \mu([x_1, x_2, \ldots, x_m])}{\log |[x_1, x_2, \ldots, x_m]|} = \alpha \right\} \text{ and}$$

$$X_\alpha := \left\{ x \in \Sigma : \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}$$

For each $q \in Q^{\complement}$, take the Gibbs state μ_q for $q\phi - T(q)\psi$. The following proposition states that the local and symbolic dimensions are equal for μ_q−typical $x \in \Sigma$. We note that Iommi and Todd (see Lemma C.1 and C.2 of [IT13]) prove this result for a class of Markov maps.

Proposition 3.5.4. *Fix* $q \in Q^{\complement}$. *Denote* μ_q *as the Gibbs state for* $q\phi - T(q)\psi$ *and* μ *as the Gibbs state for* ϕ. *Then, for* μ_q− *typical* $x = (x_1, \ldots, x_m, \ldots) \in \Sigma$,

$$\frac{\int \phi \mathrm{d}\mu_q}{-\int \psi \mathrm{d}\mu_q} = \lim_{m \to \infty} \frac{\log \mu([x_1, x_2, \ldots, x_m])}{\log |[x_1, x_2, \ldots, x_m]|} = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha(q). \tag{3.5.1}$$

Furthermore, we find that $\mu_q\left(X_\alpha^s\right) = 1$ *if* $\alpha = \alpha(q)$.

Proof. To prove that local and symbolic dimension are equal μ_q−a.e., we consider the set

$$\bar{X} := \left\{ x \in \Sigma : \lim_{m \to \infty} \frac{\psi(\sigma^m(x))}{\sum_{n=0}^{m-1} \psi(\sigma^n(x))} = 0 \right\} = \left\{ x \in \Sigma : \lim_{m \to \infty} \frac{\sum_{n=0}^{m} \psi(\sigma^n(x))}{\sum_{n=0}^{m-1} \psi(\sigma^n(x))} = 1 \right\} \tag{3.5.2}$$

because ψ is a metric potential. Then, we will prove that for any $x = (x_1, \ldots, x_m, x_{m+1}, \ldots) \in \bar{X}$,

$$\lim_{m \to \infty} \frac{\log \mu([x_1, x_2, \ldots, x_m])}{\log |[x_1, x_2, \ldots, x_m]|} = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{3.5.3}$$

and furthermore, $\mu_q(\bar{X}) = \mu_q\left(X_{\alpha(q)}^s\right) = 1$.

First, we consider cylinders and balls in Σ to prove Equation (3.5.3). For each $x = (x_1, \ldots, x_m, x_{m+1}, \ldots) \in \bar{X}$, there exists an $m \in \mathbb{N}$ large such that

$$|[x_1, \ldots, x_m, x_{m+1}]| \le r \le |[x_1, \ldots, x_m]| \tag{3.5.4}$$

for a sufficiently small $r > 0$. Then, there exists a $B(x,r) \supset |[x_1, \ldots, x_m, x_{m+1}]|$ such that

$$\mu(B(x,r)) = \mu([x_1, \ldots, x_m, x_{m+1}]). \tag{3.5.5}$$

We find that

$$\lim_{m\to\infty}\frac{\log\mu([x_1,..,x_m,x_{m+1}])}{\log|[x_1,...,x_m]|}\le\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}\le\lim_{m\to\infty}\frac{\log\mu([x_1,....,x_m,x_{m+1}])}{\log|[x_1,...,x_m,x_{m+1}]|}\tag{3.5.6}$$

by Inequality (3.5.4) and Equation (3.5.5).

We will prove that

$$\lim_{m\to\infty}\frac{\log\mu([x_1,..,x_m,x_{m+1}])}{\log|[x_1,...,x_m]|}=\lim_{m\to\infty}\frac{\log\mu([x_1,....,x_m,x_{m+1}])}{\log|[x_1,...,x_m,x_{m+1}]|}\tag{3.5.7}$$

because Inequality (3.5.6) and Equation (3.5.7) will give us Equation (3.5.3). Now, we proceed by proving that

$$\lim_{m\to\infty}\frac{\log|[x_1,...,x_m]|}{\log|[x_1,...,x_m,x_{m+1}]|}=1,\tag{3.5.8}$$

because Equation (3.5.7) will immediately follow from Equation (3.5.8).

We will use that ψ is a metric potential to prove Equation (3.5.8). Hence, for each $x=(x_1,...,x_m,x_{m+1},...)\in\bar{X}$, there exists $C>0$ such that

$$\frac{1}{C}\le\frac{|[x_1,...,x_m]|}{\prod_{n=0}^{m-1}(\exp(\psi(\sigma^n(x))))^{-1}}\le C \text{ and } \frac{1}{C}\le\frac{|[x_1,...,x_m,x_{m+1}]|}{\prod_{n=0}^{m}(\exp(\psi(\sigma^n(x))))^{-1}}\le C.\tag{3.5.9}$$

Then,

$$\frac{1}{m}[-\log C-S_m\psi(x)]\le\frac{1}{m}\log|[x_1,...,x_m]|\le\frac{1}{m}[\log C-S_m\psi(x)] \text{ and}\tag{3.5.10}$$

$$\frac{1}{m}[-\log C-S_{m+1}\psi(x)]\le\frac{1}{m}\log|[x_1,...,x_m,x_{m+1}]|\le\frac{1}{m}[\log C-S_{m+1}\psi(x)]\tag{3.5.11}$$

by Equation (3.5.9).

By Inequalities (3.5.10) and (3.5.11),

$$\lim_{m\to\infty}\frac{\frac{1}{m}[-\log C-S_m\psi(x)]}{\frac{1}{m}[\log C-S_{m+1}\psi(x)]}\le\lim_{m\to\infty}\frac{\log|[x_1,...,x_m]|}{\log|[x_1,...,x_m,x_{m+1}]|}\le\lim_{m\to\infty}\frac{\frac{1}{m}[\log C-S_m\psi(x)]}{\frac{1}{m}[-\log C-S_{m+1}\psi(x)]}.\tag{3.5.12}$$

Because $x\in\bar{X}$,

$$\lim_{m\to\infty}\frac{\log|[x_1,...,x_m]|}{\log|[x_1,...,x_m,x_{m+1}]|}=\lim_{m\to\infty}\frac{S_m\psi(x)}{S_{m+1}\psi(x)}=1.\tag{3.5.13}$$

Then,

$$\lim_{m\to\infty}\frac{\log\mu([x_1,..,x_m,x_{m+1}])}{\log|[x_1,...,x_m]|}=\lim_{m\to\infty}\frac{\log\mu([x_1,....,x_m,x_{m+1}])}{\log|[x_1,...,x_m,x_{m+1}]|}\tag{3.5.14}$$

by Inequality (3.5.13).

Hence, we find that

$$\lim_{m\to\infty}\frac{\log\mu([x_1,..,x_m,x_{m+1}])}{\log|[x_1,...,x_m]|}=\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}=\lim_{m\to\infty}\frac{\log\mu([x_1,....,x_m,x_{m+1}])}{\log|[x_1,...,x_m,x_{m+1}]|}$$

by Inequality (3.5.6) and Equation (3.5.14).

Therefore, the symbolic and local dimension of each $x = (x_1, \ldots, x_m, x_{m+1}, \ldots) \in \bar{X}$ are equal:

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \lim_{m\to\infty} \frac{\log \mu([x_1, \ldots, x_m])}{\log |[x_1, \ldots, x_m]|}. \tag{3.5.15}$$

Because μ is the Gibbs measure for ϕ, ψ is a metric potential, and $\phi, -\psi \in \mathscr{L}^1(\mu_q)$,

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \lim_{m\to\infty} \frac{\log \mu([x_1, \ldots, x_m])}{\log |[x_1, \ldots, x_m]|} = \lim_{m\to\infty} \frac{\sum_{j=0}^{m-1} \phi(\sigma^j(x))}{-\sum_{j=0}^{m-1} \psi(\sigma^j(x))} = \frac{\int \phi \mathrm{d}\mu_q}{-\int \psi \mathrm{d}\mu_q} = \alpha(q) \tag{3.5.16}$$

for each $x \in \bar{X}$ by Equation (3.5.15), Proposition 3.4.7, and the Birkhoff Ergodic Theorem. Because of the Birkhoff Ergodic Theorem,

$$\mu_q(\bar{X}) = \mu_q\left(X^s_{\alpha(q)}\right) = 1.$$

Note that Remark 4.3 of Iommi's [Iom05] states Equation (3.5.16) for μ_q−a.e. $x \in \Sigma$ for each $q \in Q^{\complement}$. Therefore, the pointwise and local dimension are equal for μ_q−typical $x \in \Sigma$. ■

We use Proposition 3.5.4 to form alternate expressions for $\alpha_{\inf}$ and $\alpha_{\sup}$. By Proposition 3.5.4 and Lemma 3.4.8, the following expressions determine the domain of the mulitfractal spectrum.

Lemma 3.5.5. *We find that*

$$\alpha_{\inf} = \inf\left\{\lim_{m\to\infty} \frac{\log \mu([x_1, \ldots, x_m])}{\log |[x_1, \ldots, x_m]|} : x = (x_1, \ldots, x_m) \in \Sigma\right\} \textit{and}$$

$$\alpha_{\sup} = \sup\left\{\lim_{m\to\infty} \frac{\log \mu([x_1, \ldots, x_m])}{\log |[x_1, \ldots, x_m]|} : x = (x_1, \ldots, x_m) \in \Sigma\right\}.$$

Proof. Assume that $d_\mu(x) = \alpha$ for some $x = (x_1, \ldots, x_m, \ldots) \in \Sigma$ and $\alpha \notin [\alpha_{\inf}, \alpha_{\sup}]$. To argue by contradiction, we will prove that there exists an invariant and ergodic Gibbs measure $\nu \in M_\sigma(\Sigma)$ such that

$$\frac{\int \phi \mathrm{d}\nu}{-\int \psi \mathrm{d}\nu} = \alpha$$

because Lemma 3.4.8 states that

$$\frac{\int \phi \mathrm{d}\nu}{-\int \psi \mathrm{d}\nu} \in [\alpha_{\inf}, \alpha_{\sup}].$$

First, we will use the definition of symbolic dimension and construct a Dirac measure. This measure will be our invariant Gibbs state.

There exists $\varepsilon > 0$ such that for each large $m \geq N$,

$$\left|\frac{\log \mu([x_1, \ldots, x_m])}{\log |[x_1, \ldots, x_m]|} - \alpha\right| \leq \varepsilon. \tag{3.5.17}$$

Now, fix an $m \geq N$ and a cylinder $[x_1,....,x_m] \subset \Sigma$ that satisfies Inequality (3.5.17). Then, there exists a $k \in \mathbb{N}$ such that

$$y = (x_1,...,x_m, y_{m+1},....,y_{m+k}, x_1,....,x_m,...) \in \Sigma \text{ and } \sigma^{m+k}(y) = y \tag{3.5.18}$$

because Σ is finitely irreducible (see Definition 2.3.14 and Lemma 2.3.15). We build a measure from this $(m+k)-$periodic sequence $y \in \Sigma$.

Consider the invariant and ergodic Gibbs measure

$$\nu = \frac{1}{m+k} \sum_{i=0}^{m+k-1} \delta_{\sigma^i(y)} \in M_\sigma(\Sigma).$$

Hence,

$$\lim_{m\to\infty} \frac{\log \nu([x_1,....,x_m, y_{m+1},...,y_{m+k}])}{\log |[x_1,...,x_m, y_{m+1},...,y_{m+k}]|} = \lim_{m\to\infty} \frac{\sum_{j=0}^{m+k-1} \phi(\sigma^j(y))}{-\sum_{j=0}^{m+k-1} \psi(\sigma^j(y))} = \frac{\int \phi \mathrm{d}\nu}{-\int \psi \mathrm{d}\nu} \tag{3.5.19}$$

because of Equation (3.5.18), ν is a Gibbs measure, $\mathscr{P}(\phi) = 0$, ψ is a metric potential, and the Birkhoff Ergodic Theorem.

There exists a sufficiently small $\delta > 0$ such that

$$\left| \frac{\sum_{j=0}^{m-1} \phi(\sigma^j(x))}{-\sum_{j=0}^{m-1} \psi(\sigma^j(x))} - \frac{\sum_{j=0}^{m+k-1} \phi(\sigma^j(y))}{-\sum_{j=0}^{m+k-1} \psi(\sigma^j(y))} \right| \leq \delta \tag{3.5.20}$$

for our sufficiently large m (that also satisfies Inequality (3.5.17)). Note that $\delta := \delta(m)$ and

$$\lim_{m\to\infty} \delta(m) = 0. \tag{3.5.21}$$

Because μ is the Gibbs measure for ϕ and ψ is a metric potential,

$$\lim_{m\to\infty} \frac{\log \mu([x_1,....,x_m])}{\log |[x_1,...,x_m]|} = \lim_{m\to\infty} \frac{\sum_{j=0}^{m-1} \phi(\sigma^j(x))}{-\sum_{j=0}^{m-1} \psi(\sigma^j(x))} \tag{3.5.22}$$

Then, we find that

$$\left| \lim_{m\to\infty} \frac{\log \nu([x_1,....,x_m, y_{m+1},...,y_{m+k}])}{\log |[x_1,...,x_m, y_{m+1},...,y_{m+k}]|} - \frac{\log \mu([x_1,....,x_m])}{\log |[x_1,...,x_m]|} \right| \leq \delta. \tag{3.5.23}$$

by Equation (3.5.19), Inequality (3.5.20), and Equation (3.5.22). Next,

$$\lim_{m\to\infty} \frac{\log \mu([x_1,....,x_m])}{\log |[x_1,...,x_m]|} = \alpha = \frac{\int \phi \mathrm{d}\nu}{-\int \psi \mathrm{d}\nu} \tag{3.5.24}$$

by Inequality (3.5.20), Equations (3.5.19), (3.5.21) and (3.5.22), and Inequality (3.5.23).

Equation (3.5.24) is a contradiction because

$$\frac{\int \phi \mathrm{d}\nu}{-\int \psi \mathrm{d}\nu} \in [\alpha_{\inf}, \alpha_{\sup}]$$

by Lemma 3.4.8 and we assumed that $\alpha \notin [\alpha_{\inf}, \alpha_{\sup}]$. Hence, for all $x = (x_1, ..., x_m, ...) \in \Sigma$,

$$\alpha_{\inf} \le \lim_{m\to\infty} \frac{\log \mu([x_1, ..., x_m])}{\log |[x_1, ..., x_m]|} \le \alpha_{\sup} \tag{3.5.25}$$

if the limit exists. Therefore,

$$\alpha_{\inf} = \inf\left\{ \lim_{m\to\infty} \frac{\log \mu([x_1, ..., x_m])}{\log |[x_1, ..., x_m]|} : x = (x_1, ..., x_m) \in \Sigma \right\} \text{ and}$$
$$\alpha_{\sup} = \sup\left\{ \lim_{m\to\infty} \frac{\log \mu([x_1, ..., x_m])}{\log |[x_1, ..., x_m]|} : x = (x_1, ..., x_m) \in \Sigma \right\}.$$

■

Because we have proven results about symbolic and pointwise dimension, we now provide the standard definition of the multifractal spectrum.

Definition 3.5.6. *For each* $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$, *the* multifractal spectrum *is the function* $f_\mu(\alpha)$ *defined by*

$$\alpha \mapsto \dim_H(X_\alpha). \tag{3.5.26}$$

By Proposition 3.5.4 and Lemma 3.5.5, we will instead use X^s_α to define the multifractal spectrum.

Proposition 3.5.7. *For each* $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$, *the* multifractal spectrum *is the function* $f_\mu(\alpha)$ *defined by*

$$\alpha \mapsto \dim_H(X^s_\alpha). \tag{3.5.27}$$

The multifractal spectrum depends on the Gibbs measure μ. We define the terms, Fenchel and Legendre transforms, because of their relevance to an equation for the multifractal spectrum.

Definition 3.5.8. *Let h be a convex function.* (h, g) *is called a* Fenchel pair *if*

$$g(p) = \sup_x \{px - h(x)\}.$$

Alternatively, we say that g is the Fenchel transform *of h. If h is a convex, twice-differentiable function, then g is called a* Legendre transform*.*

Theorem 4.1 of Iommi [Iom05] proves that the multifractal spectrum is a Legendre transform.

Theorem 3.5.9. *The multifractal spectrum f_μ is the Fenchel transform of T.*

Iommi proves that $T(q)$ is a convex function. If we take $x = q$, $h(q) = T(q)$, $h'(q) = -\alpha(q)$, and $p = -\alpha$,

$$f_\mu(\alpha) = g(p) = \sup_{x\in\mathbb{R}}\{px - h(x)\} = \sup_{q\in\mathbb{R}}\{-q\alpha - T(q)\} = \inf_{q\in\mathbb{R}}\{T(q) + q\alpha\}.$$

Hence, (T, f_μ) form a Fenchel pair. Thus, we will use the following form of Iommi's theorem to prove Theorem 3.1.5.

Theorem 3.5.10. *For each* $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$,

$$f_\mu(\alpha) = \inf_{q\in\mathbb{R}}\{T(q) + q\alpha\}.$$

Assume that Q is a closed interval. Then, Proposition 3.4.3 states that $T(q)$ has at most two phase transitions. Using Theorem 3.5.10, we expect the multifractal spectrum to have phase transitions.

3.6 The Multifractal Spectrum's Analytic Regions

Before we prove that it is possible for the multifractal spectrum to have phase transitions, we find its analytic regions. We will combine our results about the analytic regions of $T(q)$ (see Proposition 3.4.4) and $\alpha(q)$ (see Propositions 3.4.6) to prove Theorems 3.1.5. We will use Proposition 4.7 of [Iom05] to prove Theorem 3.1.6. First, we recall essential functions, sets, and results. For each $q \in \mathbb{R}$, the temperature function

$$T(q) := \inf\{t \in \mathbb{R} : \mathscr{P}(q\phi - t\psi) \le 0\}$$

and the function

$$\tilde{t}(q) := \inf\{t \in \mathbb{R} : \mathscr{P}(q\phi - t\psi) < \infty\}.$$

Consider the sets

$$Q := \{q \in \mathbb{R} : T(q) = \tilde{t}(q)\} \text{ and } Q^{\complement} := \{q \in \mathbb{R} : T(q) > \tilde{t}(q)\}.$$

Recall that $T(q)$ and $\alpha(q)$ are analytic on open sub-intervals of Q and $Q^{\complement}$ (see Propositions 3.4.3 and 3.4.6).

Without loss of generality, assume that there exist $0 < q_0 < q_1 < \infty$ such that $Q = [q_0, q_1]$. Denote μ as the Gibbs state for ϕ. By Lemma 3.5.5 and Equation (3.4.6),

$$(\alpha_{\inf}, \alpha_{\sup}) = (\alpha_{\inf}, \alpha^+) \cup \{\alpha^+\} \cup (\alpha^+, \alpha_{\lim}) \cup \{\alpha_{\lim}\} \cup (\alpha_{\lim}, \alpha^-) \cup \{\alpha^-\} \cup (\alpha^-, \alpha_{\sup}). \tag{3.6.1}$$

We will use Equation (3.6.1) to find the multifractal spectrum's analytic regions and phase transitions. We outline the steps for proving that the multifractal spectrum can have phase transitions. Using Propositions 3.4.3 and 3.4.6,

I we prove that the multifractal spectrum is analytic on $(\alpha_{\inf}, \alpha^+)$ and $(\alpha^-, \alpha_{\sup})$,

II assuming that $0 < \alpha_{\lim} < \infty$, we prove that the multifractal spectrum is analytic on $(\alpha^+, \alpha_{\lim})$ and $(\alpha_{\lim}, \alpha^-)$, and

III we finally combine our results to prove Theorem 3.1.5 and 3.1.6.

The analytic regions of the multifractal spectrum will help us find its non-analytic points.

3.6.1 The Multifractal Spectrum on $(\alpha_{\inf}, \alpha^+)$ and $(\alpha^-, \alpha_{\sup})$

We will prove that the multifractal spectrum is analytic on $(\alpha_{\inf}, \alpha^+)$ and $(\alpha^-, \alpha_{\sup})$.

Lemma 3.6.1. *For each α in an open sub-interval $S \subset \left\{\alpha(q) : q \in Q^{\complement}\right\}$,*

$$f_\mu(\alpha) = f_\mu(\alpha(q)) = T(q) + q\alpha(q). \tag{3.6.2}$$

Proof. Fix an α in an open sub-interval $S \subset \left\{\alpha(q) : q \in Q^{\complement}\right\}$. Hence, there exists a q, in an open sub-interval $P \subset Q^{\complement}$, such that $\alpha = \alpha(q)$. By Theorem 3.5.10,

$$f_\mu(\alpha) = \inf_{q\in\mathbb{R}}\{T(q) + q\alpha\}. \tag{3.6.3}$$

Because of Equation (3.6.3) and $T(q)$ is analytic on $P \subset Q^{\complement}$,

$$\frac{\mathrm{d}}{\mathrm{d}q}(T(q) + q\alpha) = T'(q) + \alpha = 0 \tag{3.6.4}$$

when $\alpha = -T'(q) = \alpha(q)$. Thus,

$$f_\mu(\alpha) = T(q) + q\alpha(q)$$

by Equation (3.6.4). ■

This lemma helps us find a region where the multifractal spectrum is analytic.

Proposition 3.6.2. *The multifractal spectrum f_μ is analytic and concave on any open sub-interval S of $\left\{\alpha(q) : q \in Q^{\complement}\right\}$.*

Proof. First, we prove analyticity. Take an α in an open sub-interval $S \subset \left\{\alpha(q) : q \in Q^{\complement}\right\}$. Hence, there exists a q, in an open sub-interval $P \subset Q^{\complement}$, such that $\alpha = \alpha(q)$. By Equation (3.6.2), $f_\mu(\alpha(q)) = T(q) + q\alpha(q)$.

To prove that the multifractal spectrum is analytic on S, we will use that $T(q)$ and $\alpha(q)$ are analytic on P by Propositions 3.4.3 and 3.4.6. Now, we take the derivative of $f_\mu(\alpha(q))$:

$$\frac{\mathrm{d}}{\mathrm{d}q} f_\mu(\alpha(q)) = \frac{d}{d\alpha(q)}(f_\mu(\alpha(q)))\alpha'(q) = q\alpha'(q). \tag{3.6.5}$$

Then, the derivative of the multifractal spectrum with respect to $\alpha(q)$ is

$$\frac{\mathrm{d}}{\mathrm{d}\alpha(q)}(f_\mu(\alpha(q))) = q. \tag{3.6.6}$$

Because we took the derivative in terms of $\alpha(q)$, q is a function of α, i.e., $q = q(\alpha)$. Then, $-T''(q) < 0$ for each $q \in P$ because $T(q)$ is convex on P. Because $\alpha'(q) = -T''(q) < 0$, $\alpha(q)$ and $q(\alpha)$ are invertible. Hence, because $\alpha(q)$ is analytic on P, the function $\frac{\mathrm{d}}{\mathrm{d}\alpha(q)}(f_\mu(\alpha(q))) = q(\alpha)$ is analytic on S. Thus, as $T(q(\alpha))$ and $q(\alpha)$ are analytic on S, $f_\mu(\alpha)$ is analytic for each α in $S \subset \left\{\alpha(q) : q \in Q^{\complement}\right\}$.

To prove that $f_\mu(\alpha)$ is concave on an open sub-interval S of $\left\{\alpha(q) : q \in Q^{\complement}\right\}$, we take additional derivatives of f_μ with respect to q. By Equation (3.6.6),

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}(f_\mu(\alpha(q))) = q'(\alpha) < 0$$

because $T(q)$ is convex. Thus, the multifractal spectrum is concave on any open $S \subset \left\{\alpha(q) : q \in Q^{\complement}\right\}$. ■

From Proposition 3.6.2, we are now able to prove that the multifractal spectrum is analytic on two sub-intervals of $(\alpha_{\mathrm{inf}}, \alpha_{\mathrm{sup}})$.

Proposition 3.6.3. *The multifractal spectrum f_μ is analytic and concave on $(\alpha_{\mathrm{inf}}, \alpha^+)$ and $(\alpha^-, \alpha_{\mathrm{sup}})$.*

Proof. Take an arbitrary $\alpha \in (\alpha_{\mathrm{inf}}, \alpha^+) \cup (\alpha^-, \alpha_{\mathrm{sup}})$. There exists a unique $q \notin [q_0, q_1]$ such that $\alpha = \alpha(q)$. Hence, $(\alpha_{\mathrm{inf}}, \alpha^+)$ and $(\alpha^-, \alpha_{\mathrm{sup}})$ are open sub-intervals of $\left\{\alpha(q) : q \in Q^{\complement}\right\}$. The result follows from Proposition 3.6.2. ■

The proof of Proposition 3.6.2 leads to result about the increasing and decreasing behaviour of the multifractal spectrum.

Proposition 3.6.4. *The multifractal spectrum f_μ*

1. *increases on open sub-intervals S of $\{\alpha(q) : q > 0\} \cap \left\{\alpha(q) : q \in Q^{\complement}\right\}$*
2. *decreases on open sub-intervals of $\{\alpha(q) : q < 0\} \cap \left\{\alpha(q) : q \in Q^{\complement}\right\}$.*

Proof. Consider an open sub-interval $S \subset \left\{\alpha(q) : q \in Q^{\complement}\right\}$. Take an arbitrary $\alpha \in S$. Equation (3.6.5) states that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(f_\mu(\alpha)) = \frac{\mathrm{d}}{\mathrm{d}\alpha}(f_\mu(\alpha(q)) = q. \tag{3.6.7}$$

The result follows from Equation (3.6.7). ■

In summary, we proved that if $T(q)$ is analytic on open sub-intervals of $Q^{\complement}$, then $\alpha(q)$ is analytic on such sub-intervals. In turn, the multifractal spectrum f_μ is analytic on open sub-intervals of $\{\alpha(q) : q \in Q^{\complement}\}$. We also found a result about the increasing or decreasing behaviour of f_μ on these sub-intervals.

3.6.2 The Multifractal Spectrum on $(\alpha^+, \alpha_{\text{lim}})$ and $(\alpha_{\text{lim}}, \alpha^-)$

Now, we prove that the multifractal spectrum is analytic on $(\alpha^+, \alpha_{\text{lim}})$ and $(\alpha_{\text{lim}}, \alpha^-)$. Recall that

$$\alpha^- := \lim_{q \to q_0^-} -T'(q) \text{ and } \alpha^+ := \lim_{q \to q_1^+} -T'(q). \tag{3.6.8}$$

Proposition 3.6.5. *The multifractal spectrum*

$$f_\mu(\alpha) = \begin{cases} T(q_1) + q_1\alpha & \text{on } (\alpha^+, \alpha_{\text{lim}}) \\ T(q_0) + q_0\alpha & \text{on } (\alpha_{\text{lim}}, \alpha^-) \end{cases}$$

and is analytic on each of these intervals. In particular, the multifractal spectrum is an increasing linear function on $(\alpha^+, \alpha_{\text{lim}})$ and $(\alpha_{\text{lim}}, \alpha^-)$ if $Q = [q_0, q_1]$ for $0 < q_0 < q_1 < \infty$.

Proof. Recall that $Q = [q_0, q_1]$ for $0 < q_0 < q_1 < \infty$. Fix $\alpha \in (\alpha^+, \alpha_{\text{lim}})$. Then,

$$f_\mu(\alpha) = \inf_{q \in \mathbb{R}} \{T(q) + q\alpha\} = T(q_1) + q_1\alpha$$

by Equation (3.6.8). This gives us that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} f_\mu(\alpha) = q_1 > 0.$$

Thus, $f_\mu(\alpha)$ is an increasing linear function with slope q_1 on $(\alpha^+, \alpha_{\text{lim}})$.

Now, fix $\alpha \in (\alpha_{\text{lim}}, \alpha^-)$. Then,

$$f_\mu(\alpha) = \inf_{q \in \mathbb{R}} \{T(q) + q\alpha\} = T(q_0) + q_0\alpha$$

by Equation (3.6.8). Hence,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} f_\mu(\alpha) = q_0 > 0.$$

Thus, $f_\mu(\alpha)$ is an increasing linear function with slope q_0 on the interval $(\alpha_{\text{lim}}, \alpha^-)$. ∎

With this proposition in mind, we can finally prove Theorems 3.1.5 and 3.1.6.

3.7 The Multifractal Spectrum's Phase Transitions

Given our results about the multifractal spectrum's analytic regions (see Propositions 3.6.3 and 3.6.5), we will now prove that the multifractal spectrum has 0 to 3 phase transitions when $0 < \alpha_{\text{lim}} < \infty$ and 0 to 1 phase transition when $\alpha_{\text{lim}} = \infty$. As stated on Proposition 3.4.2, Q is either a closed interval, a half-open infinite interval, a point, or the empty set when $0 < \alpha_{\text{lim}} < \infty$. To prove Theorem 3.1.5, we must consider the behaviour of the multifractal spectrum for each possible form of Q.

3.7.1 Positive Closed Interval

First, we assume that $Q = [q_0, q_1]$ such that $0 < q_0 < q_1 < \infty$.

Proposition 3.7.1. *Assume that (Σ, σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to ψ. Denote μ as the Gibbs state for ϕ. The multifractal spectrum behaves four different ways (respectively called cases 1-4) as follows.*

1. *If $\alpha^- > \alpha_{\lim} > \alpha^+$, then the function f_μ is concave and analytic on $(\alpha_{\inf}, \alpha^+)$, $(\alpha^+, \alpha_{\lim})$, $(\alpha_{\lim}, \alpha^-)$, and $(\alpha^-, \alpha_{\sup})$ and there exist phase transitions at $\alpha^-, \alpha_{\lim}$, and α^+.*
2. *If $\alpha^- > \alpha_{\lim} = \alpha^+$, then the function f_μ is concave and analytic on $(\alpha_{\inf}, \alpha_{\lim})$, $(\alpha_{\lim}, \alpha^-)$, and $(\alpha^-, \alpha_{\sup})$ and there exist phase transitions at α^- and $\alpha_{\lim}$.*
3. *If $\alpha^- = \alpha_{\lim} > \alpha^+$, then the function f_μ is concave and analytic on $(\alpha_{\inf}, \alpha^+)$, $(\alpha^+, \alpha_{\lim})$, and $(\alpha_{\lim}, \alpha_{\sup})$ and there exist phase transitions at α^+ and $\alpha_{\lim}$.*
4. *If $\alpha^- = \alpha_{\lim} = \alpha^+$, then the function f_μ is concave and analytic on $(\alpha_{\inf}, \alpha_{\lim})$ and $(\alpha_{\lim}, \alpha_{\sup})$ and there exists a possible phase transition at $\alpha_{\lim}$.*

Proof. We only prove case 1 because cases 2 to 4 are similarly proven. We use Equation (3.6.1):

$$(\alpha_{\inf}, \alpha_{\sup}) = (\alpha_{\inf}, \alpha^+) \cup \{\alpha^+\} \cup (\alpha^+, \alpha_{\lim}) \cup \{\alpha_{\lim}\} \cup (\alpha_{\lim}, \alpha^-) \cup \{\alpha^-\} \cup (\alpha^-, \alpha_{\sup}).$$

Hence, by Propositions 3.6.3 and 3.6.5, the multifractal spectrum is analytic and concave on $(\alpha_{\inf}, \alpha^+) \cup (\alpha^+, \alpha_{\lim}) \cup (\alpha_{\lim}, \alpha^-) \cup (\alpha^-, \alpha_{\sup})$. Furthermore, f_μ is linear on $(\alpha^+, \alpha_{\lim}) \cup (\alpha_{\lim}, \alpha^-)$. Therefore, there exist phase transitions at α^+, $\alpha_{\lim}$, and α^-. ■

3.7.2 Other Intervals

Now, we consider every possible form for $Q = [q_0, q_1]$ such that $-\infty \le q_0 \le q_1 \le \infty$. Proving such behaviour follows from the same techniques used to prove Proposition 3.7.1.

Proposition 3.7.2. *The multifractal spectrum has varying numbers of phase transitions*

1. *If Q is a closed interval, then the multifractal spectrum has 0 to 3 phase transitions.*
2. *If Q is a point in the reals, then $\alpha^+ \le \alpha_{\lim} \le \alpha^-$. Hence, the multifractal spectrum has 0 to 3 phase transitions.*
3. *If Q is the half-open interval $(-\infty, q_1]$, then $\alpha_{\lim} \ge \alpha^+$. The multifractal spectrum has 0 to 1 phase transition.*
4. *If Q is the half-open interval $[q_0, \infty)$, then $\alpha^- \ge \alpha_{\lim}$. The multifractal spectrum has 0 to 1 phase transition.*

5. *If Q is the open interval $(-\infty,\infty)$, then $\alpha^+ = \alpha^- = \alpha_{\lim} = \alpha_{\inf} = \alpha_{\sup}$. The multifractal spectrum $f_\mu(\alpha) = t_\infty$ on all of its domain.*

Proposition 3.7.3. *Assume that $Q = \emptyset$. Then, the multifractal spectrum has no phase transitions.*

Proof. Because $Q = \emptyset$, $T(q) > \tilde{t}(q)$ for all $q \in \mathbb{R}$. Fix an arbitrary $q \in \mathbb{R}$. Then, $t \mapsto \mathscr{P}(q\phi - t\psi)$ is analytic on $(T(q)-\varepsilon, T(q)+\varepsilon)$ any sufficiently small $\varepsilon > 0$ by Proposition 3.3.12. This gives us that $T(q)$ is analytic on all of $\mathbb{R}$ by Proposition 3.4.4. Therefore, the proposition follows from Proposition 3.6.2. ■

We will restate and finally, prove Theorems 3.1.5 and 3.1.6.

3.8 Proving Our Main Results

We revisit our main results on the multifractal spectrum's phase transitions.

Theorem 3.8.1. *Assume that (Σ,σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to $-\psi$ and the potentials were chosen so that $0 < \alpha_{\lim} < \infty$. Denote μ as the Gibbs state for ϕ.*

1. *There exist intervals A_i such that $f_\mu(\alpha)$ is analytic on each of their interiors.*
2. *The interval $(\alpha_{\inf}, \alpha_{\sup}) = \cup_{i=1}^{j} A_i$ such that $j \in \{1,2,3,4\}$.*
3. *The multifractal spectrum is concave on $(\alpha_{\inf}, \alpha_{\sup})$, has its maximum at a single point, and has zero to three phase transitions.*

Proof. The result follows from Propositions 3.7.1, 3.7.2, and 3.7.3. ■

We now analyse the multifractal spectrum's behaviour when $\alpha_{\lim} = \infty$. Iommi [Iom05] has already analysed a generalisation of this case.

Theorem 3.8.2. *Assume that (Σ,σ) satisfies the BIP property and is topologically mixing. Let $\phi : \Sigma \to \mathbb{R}^-$ be a locally Hölder potential such that $\mathscr{P}(\phi) = 0$ and $\psi : \Sigma \to \mathbb{R}^+$ be a metric potential such that $\mathscr{P}(-\psi) < \infty$. Assume that ϕ is non-cohomologous to $-\psi$ and the potentials were chosen so that $\alpha_{\lim} = \infty$. Denote μ as the Gibbs state for ϕ.*

1. *There exist intervals A_i such that $f_\mu(\alpha)$ is analytic on each of their interiors.*
2. *The interval $(\alpha_{\inf}, \alpha_{\sup}) = \cup_{i=1}^{j} A_i$ such that $j \in \{1,2\}$.*
3. *The multifractal spectrum is concave on $(\alpha_{\inf}, \alpha_{\sup})$ and has zero to one phase transition.*

Proof. Because $\alpha_{\lim} = \infty$, the multifractal spectrum has unbounded domain. Define

$$q^* := \inf\{q \in \mathbb{R} : \exists t \in \mathbb{R} \textit{ satisfying } \mathscr{P}(q\phi - t\psi) \le 0\}$$

as given by Definition 4.1 of Iommi [Iom05]. Then, $q^* = 0$ by Proposition 4.5 of Iommi [Iom05]. Thus, by Proposition 4.6 of Iommi [Iom05], the multifractal spectrum is analytic or there exists a phase transition on $\alpha(0)$. ■

Next, we consider a map to form examples of phase transitions for f_μ.

3.9 Adaptation For the Gauss Map

Consider the dynamical system $(G, [0,1] \setminus \mathbb{Q})$ given by the Gauss map G and its shift space (Σ, σ). We will take a Gibbs measure on this shift space. We briefly will discuss the characteristics of this map that allow us to apply Theorems 3.1.5 and 3.1.6 and then, form examples of phase transitions for the multifractal spectrum. First, we define the Gauss map and analyse its dynamics.

3.9.1 The Gauss Map

First, we state the definition of the Gauss map.

Definition 3.9.1. *The* Gauss map $G : [0,1] \setminus \mathbb{Q} \to [0,1] \setminus \mathbb{Q}$ *is*

$$G(x) = \frac{1}{x} \mod 1$$

for each $x \in [0,1] \setminus \mathbb{Q}$.

We also define the inverse branches of the Gauss map.

Definition 3.9.2. *Let* $I_b := \left(\frac{1}{b+1}, \frac{1}{b}\right] \setminus \mathbb{Q}$ *for each* $b \in \mathbb{N}$. *Consider the Gauss map* G. *Define* $G_b : [0,1] \setminus \mathbb{Q} \to I_b$, *the* inverse branch $(G|_{I_b})^{-1}$ *of the Gauss map, as*

$$G_b(x) = \frac{1}{x+b}$$

for $x \in [0,1] \setminus \mathbb{Q}$. *Let* $n \in \mathbb{N}$. *For each* $\tilde{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$, *the composition* $G_{\tilde{b}}$ *of these inverse branches is*

$$G_{\tilde{b}} := G_{b_1} \circ G_{b_2} \circ \cdots \circ G_{b_n}.$$

From this point, we use the full shift $\Sigma = \mathbb{N}^{\mathbb{N}}$. The Gauss map has a conjugacy (up to a countable number of points) with the left shift σ. The coding map between Σ and $[0,1]$ is the continued fraction map (see Khinchin [KE64]).

Definition 3.9.3. *Take any $i, b \in \mathbb{N}$. Consider the Gauss map $G : [0,1] \setminus \mathbb{Q} \to [0,1] \setminus \mathbb{Q}$. For each sequence $a = (a_1(x), a_2(x), \ldots) \in \Sigma$, there exists an $x \in [0,1] \setminus \mathbb{Q}$. The* coding map $\pi : \Sigma \to [0,1] \setminus \mathbb{Q}$ *between each $a \in \Sigma$ and $x \in [0,1] \setminus \mathbb{Q}$ is given by*

$$\pi(a) := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}}$$

such that

1. $x = \pi(a)$
2. *if $G^{i-1}(x) \in \left(\frac{1}{b+1}, \frac{1}{b}\right]$, then $a_i(x) = b$.*

3.9.2 Conditions for Markov Maps

We briefly discuss applying our main results, Theorems 3.1.5 and 3.1.6, to a class of expanding Markov maps $T : (0,1) \to (0,1)$ and their respective shifts (Σ, σ). The expanding map must satisfy the following conditions to apply our theorems. Denote $|S|$ as the diameter of a set $S \subset \Sigma_A$. The expanding map T have a countable Markov partition $\{R_1, R_2, \ldots, R_m, \ldots\}$, with $R_{x_1,\ldots,x_m} := R_{x_1} \cap T^{-1}(R_{x_2}) \cap \cdots \cap T^{-(m-1)}(R_{x_m})$, such that

1. $\lim\limits_{m\to\infty} R_m = \{0\}$ and
2. $\lim\limits_{m\to\infty} \frac{\log|R_{x_1,\ldots,x_m}|}{\log|R_{x_1,\ldots,x_m,x_{m+1}}|} = 1$.

Furthermore, the map must satisfy the big images and pre-images (BIP) property and be topologically mixing (defined as follows).

Definition 3.9.4. *Let $X \subset R$ be an interval. A countably-branched Markov map $T : X \to X$ satisfies the* big images and pre-images property *if there exists a finite subset $\{a_1, \ldots, a_n\}$ of its Markov partition α such that for any $a \in \alpha$, there exist $i, j \in \{1, \ldots, n\}$ such that $a \subset T(a_i)$ and $T^{-1}(a_j) \subset a$ (modulo 0).*

We now define the notion of topological mixing for Markov maps.

Definition 3.9.5. *Let $X \subset R$ be an interval. A topological dynamical system $T : X \to X$ is* topologically mixing *if for any two open non-empty sets $U, V \subset X$, there exists $N = N(U,V) \in \mathbb{N}$ such that for all $n > N$,*

$$T^n(U) \cap V \neq \emptyset.$$

However, we will solely concentrate on applying our results to the countable full shift (Σ, σ). As stated earlier, the Gauss map is modelled by a countable full Markov shift that satisfies the BIP property and is topologically mixing. We will later consider the definition of the multifractal spectrum with respect to a measure on $[0,1]$ (see Definition 3.10.4). Now, we develop results on the necessary thermodynamic formalism for our multifractal analysis.

3.9.3 Thermodynamic Formalism

Now, we define the potentials ψ and ϕ used in our multifractal analysis.

Definition 3.9.6. *Define the locally Hölder potential $\psi : \Sigma \to \mathbb{R}^+$ on each sequence $x \in \Sigma$ by*

$$\psi(x) = \log|G'(\pi(x))|.$$

Also, take a locally Hölder potential $\phi : \Sigma \to \mathbb{R}^-$ such that $\mathscr{P}(\phi) = 0$ and it is non-cohomologous to ψ. Denote μ as the Gibbs measure for ϕ. We also consider the measure $\bar{\mu} = \mu \circ \pi^{-1}$ on $[0,1]$.

We prove that ψ is integrable and a metric potential, so that we can apply Theorems 3.1.5 and 3.1.6 to our examples in Section 3.11. We assume that $Q := [q_0, q_1]$ for fixed $0 < q_0 < q_1 < \infty$, so $\psi \in \mathscr{L}^1(\mu)$ by Proposition 3.3.15 (as $0 \in Q^\complement$).

Lemma 3.9.7. *The function ψ is a metric potential. Furthermore,*

$$\lim_{m\to\infty} -\frac{1}{m}\sum_{i=0}^{m-1} \log|G'(G^i(z))| = -\int_\Sigma \psi \, \mathrm{d}\mu = -\int_0^1 \log|G'| \, \mathrm{d}\bar{\mu} = \lim_{m\to\infty} \frac{1}{m} \log|[x_1,..,x_m]| \tag{3.9.1}$$

for $\bar{\mu}$–typical $z = \pi(x) \in [0,1] \setminus \mathbb{Q}$ such that $x = (x_1, x_2, ..., x_m, ...) \in \Sigma$.

Proof. Take an arbitrary $[x_1,...,x_m] \subset \Sigma$. By the mean value theorem, there exists a $z \in \pi([x_1,...,x_m])$ such that

$$|(G^m)'(z)| = (|[x_1,...,x_m]|)^{-1}. \tag{3.9.2}$$

Also,

$$(G^m)'(z) = G'(G^{m-1}(z))G'(G^{m-2}(z)) \cdots G'(z). \tag{3.9.3}$$

By Equations (3.9.2) and (3.9.3),

$$\sum_{i=0}^{m-1} \psi(\sigma^i(x)) = \log|(G^m)'(z)| = -\log|[x_1,...,x_m]| = \sum_{i=0}^{m-1} \log|G'(G^i(z))|. \tag{3.9.4}$$

Then, there exists an $x = (x_1,...,x_m,...) \in \Sigma$ such that

$$|[x_1,...,x_m]| = \exp\left(-\sum_{i=0}^{m-1} \psi(\sigma^i(x))\right) = \prod_{i=0}^{m-1} (\exp \psi(\sigma^i(x)))^{-1},$$

so ψ is a metric potential. Therefore,

$$\lim_{m\to\infty} -\frac{1}{m}\sum_{i=0}^{m-1} \log|G'(G^i(z))| = -\int_\Sigma \psi \, \mathrm{d}\mu = -\int_0^1 \log|G'| \, \mathrm{d}\bar{\mu} = \lim_{m\to\infty} \frac{1}{m} \log|[x_1,..,x_m]|$$

for $\bar{\mu}$–typical $z = \pi(x) \in [0,1] \setminus \mathbb{Q}$, such that $x = (x_1, x_2, ..., x_m, ...) \in \Sigma$, by Equation (3.9.4) and the Birkhoff Ergodic Theorem. ∎

Besides our potentials, our analysis of the multifractal spectrum depends on level sets defined by symbolic and local dimension (see Definitions 3.10.1 and 3.10.3).

3.10 The Multifractal Spectrum and Level Sets

Again, consider the Gibbs measure μ for $\phi : \Sigma \to \mathbb{R}^-$, the measure $\bar{\mu} = \mu \circ \pi^{-1}$ on $[0,1]$, and the values $\alpha_{\inf}$ and $\alpha_{\sup}$ (see Lemma 3.5.5). In this section, we will justify our choice to analyse the multifractal spectrum with respect to μ rather than $\bar{\mu}$. We will state the typical definition of the multifractal spectrum for $\bar{\mu}$, but we first revisit the notion of the multifractal spectrum for μ. Recall the definition of symbolic dimension and the level set X^s_α.

Definition 3.10.1. *The* symbolic dimension *of a sequence* $x = (x_1, x_2, ..., x_m, ...) \in \Sigma$ *is*

$$d_\mu(x) := \lim_{m\to\infty} \frac{\log \mu([x_1, x_2, ..., x_m])}{\log |[x_1, x_2, ..., x_m]|}.$$

We take the set

$$X^s_\alpha := \left\{ x \in \Sigma : \lim_{m\to\infty} \frac{\log \mu([x_1, x_2, ..., x_m])}{\log |[x_1, x_2, ..., x_m]|} = \alpha \right\}$$

for each fixed $\alpha \in [\alpha_{\inf}, \alpha_{\sup}]$.

Definition 3.10.2. *The* multifractal spectrum with respect to μ *is the function*

$$f_\mu : \alpha \mapsto \dim_H(X^s_\alpha)$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$.

We proved results about the number of non-analytic points or phase transitions of this function (see Theorems 3.1.5 and 3.1.6). Typically, the multifractal spectrum $f_{\bar{\mu}}$ is defined by $\bar{\mu}$ and level sets on $[0,1]$. We will now reintroduce this notion of the multifractal spectrum.

To introduce the function $f_{\bar{\mu}}$, we consider the local dimension of each $x \in [0,1]$ and define the level sets used to define it.

Definition 3.10.3. *The* local dimension *of* $x \in [0,1]$ *is*

$$d_{\bar{\mu}}(x) = \lim_{r\to 0} \frac{\log \bar{\mu}(B(x,r))}{\log r}.$$

Define the level set

$$X_\alpha = \left\{ x \in [0,1] \setminus \mathbb{Q} : \lim_{r\to 0} \frac{\log \bar{\mu}(B(x,r))}{\log r} = \alpha \right\}$$

for each $\alpha \in [\alpha_{\inf}, \alpha_{\sup}]$.

Definition 3.10.4. *The* multifractal spectrum with respect to $\bar{\mu}$ *is the function*

$$f_{\bar{\mu}} : \alpha \mapsto \dim_H(X_\alpha)$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$.

Take the projection of the set X_α^s :

$$\pi(X_\alpha^s) = \pi\left(\left\{x \in \Sigma : \lim_{m\to\infty} \frac{\log\mu([x_1,...,x_m])}{\log|[x_1,...,x_m]|} = \alpha\right\}\right).$$

We find that

$$\dim_H(X_\alpha^s) = \dim_H(\pi(X_\alpha^s))$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$ and state a key result that follows from Iommi [Iom05] (Page 1891, Theorem 3.7).

Proposition 3.10.5. *The multifractal spectrum*

$$f_\mu(\alpha) = \inf_{q\in\mathbb{R}}\{T(q) + q\alpha\} = \dim_H(\pi(X_\alpha^s))$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$.

We prove that both multifractal spectra are equivalent (see Definitions 3.10.2 and 3.10.4).

Proposition 3.10.6. *Consider the locally Hölder potential* $\psi = \log|G' \circ \pi|$ *and a locally Hölder potential* $\phi : \Sigma \to \mathbb{R}^-$ *such that* $\mathscr{P}(\phi) = 0$ *and it is non-cohomologous to* ψ*. Denote* μ *as the Gibbs state for* ϕ *and* $\bar{\mu} = \mu \circ \pi^{-1}$*. Then,*

$$f_\mu(\alpha) = f_{\bar{\mu}}(\alpha)$$

for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$.

We outline the proof to Proposition 3.10.6. Using Definition 3.10.4, we proceed by proving that $\dim_H(X_\alpha) \geq f_\mu(\alpha)$ and $\dim_H(X_\alpha) \leq f_\mu(\alpha)$ for each $\alpha \in (\alpha_{\inf}, \alpha_{\sup})$.

To prove $\dim_H(X_\alpha) \geq f_\mu(\alpha)$, we consider a sequence of finite state shifts $\Sigma_n := \{1,...,n\}^{\mathbb{N}}$. Then, we define the level sets

$$X_{\alpha,n}^s := \left\{x \in \Sigma_n : \lim_{m\to\infty} \frac{\log\mu([x_1,x_2,...,x_m])}{\log|[x_1,x_2,...,x_m]|} = \alpha\right\} \text{ and}$$

$$X_{\alpha,n} = \left\{x \in \pi(\Sigma_n) : \lim_{r\to 0} \frac{\log\bar{\mu}(B(x,r))}{\log r} = \alpha\right\}.$$

We would prove that $\dim_H(X_{\alpha,n}^s) \leq \dim_H(X_{\alpha,n})$. To do this, we would use the compact approximation of pressure, results on the mulitfractal analysis for countable Markov shifts from Iommi [Iom05], thermodynamic formalism for finite state Markov shifts by Ruelle [Rue04], and multifractal analysis for finite state Markov shifts by Pesin and Weiss [PW97].

To prove that $\dim_H(X_\alpha) \leq f_\mu(\alpha)$, we again use the behaviour of the Gauss map. For a sufficiently large $n \in \mathbb{N}$, we would consider a cylinder $[x_1,...,x_n]$ that contains an element $x \in X_\alpha^s$. Then, we would prove that either $[x_1,...,x_n - 1]$ or $[x_1,...,x_n + 1]$ contains an element $y \in \pi^{-1}(X_\alpha)$. Finally, we use this result to create a Hausdorff cover for $\pi^{-1}(X_\alpha)$. This leads to the upper bound.

Given Proposition 3.10.6, we have justified our choice to analyse the behaviour of the multifractal spectrum f_μ with respect to a measure on Σ instead of $f_{\bar{\mu}}$. Now, we construct examples of non-analytic points or phase transitions for the multifractal spectrum.

3.11 Examples of Phase Transitions for the Multifractal Spectrum

We will first construct examples that apply the results of Theorem 3.1.5 and later, we will form examples to apply Theorem 3.1.6. Consider the Gauss map $G:[0,1]\setminus\mathbb{Q}\to[0,1]\setminus\mathbb{Q}$ and the countable full shift (Σ,σ). Take the continued fraction map $\pi:\Sigma\to[0,1]\setminus\mathbb{Q}$ and metric potential $\psi:\Sigma\to\mathbb{R}^+$ given by $\psi(x)=\log|G'(\pi(x))|$. We will also consider a locally constant potential $\phi:\Sigma\to\mathbb{R}^-$ that meets the conditions of Theorem 3.1.5.

However, we will instead use locally constant potentials for our examples (see the discussion of Inequality (3.11.2)). We will estimate ψ with one locally constant potential $\tilde{\psi}$ in all of our examples and we will define a locally constant potential $\tilde{\phi}:\Sigma\to\mathbb{R}^-$ in each of our examples. The next subsection provides the locally constant potentials used in our examples.

3.11.1 Locally Constant Potentials

We will now state results for the locally constant potentials used in our examples. Proposition 2.3.19 states that locally Hölder functions can be approximated by using locally constant functions. First, we define the construction of these functions and explain how approximation applies to ψ. Let $\sum_{i=1}^{\infty}p_i=1$ such that $p_i>0$ and $\sum_{i=1}^{\infty}s_i=1$ such that $s_i>0$ for each $i\in\mathbb{N}$. For all of our examples, we will define

$$s_i=\frac{6}{\pi^2 i^2} \tag{3.11.1}$$

for each $i\in\mathbb{N}$.

For each $x=(x_1,x_2,x_3,\ldots)\in\Sigma$, we consider $\tilde{\phi}:\Sigma\to\mathbb{R}^-$ and $\tilde{\psi}:\Sigma\to\mathbb{R}^+$ respectively given by

$$\tilde{\phi}(x)=\log p_{x_1} \text{ and } \tilde{\psi}(x)=\log s_{x_1}^{-1}.$$

Hence, changing how we define p_i for all of our examples will lead to various possible phase transitions for the multifractal spectrum. Because of Equation (3.11.1), ψ and $\tilde{\psi}$ are locally Hölder, and $G'(z)=\frac{1}{z^2}$ for each $z\in[0,1]\setminus\mathbb{Q}$, there exists a sufficiently small $\varepsilon>0$ such that

$$|\tilde{\psi}(x)-\psi(x)|\le\varepsilon \tag{3.11.2}$$

for each $x\in\Sigma$.

We now form results about pressure and $\alpha_{\lim}$. Then, because $\tilde{\psi}$ and $\tilde{\phi}$ are defined by the first symbol of each $x\in\Sigma$,

$$\mathscr{P}(q\tilde{\phi}-t\tilde{\psi})=\log\left(\sum_{i=1}^{\infty}p_i^q s_i^t\right). \tag{3.11.3}$$

Consider an arbitrary sequence $\hat{i}=(i,y_2,y_3,\ldots)\in\Sigma$ for each $i\in\mathbb{N}$ and the sequence $\bar{i}=(i,i,i,\ldots)\in\Sigma$ for each $i\in\mathbb{N}$. Because $\tilde{\psi}$ approximates ψ (see Inequality (3.11.2)),

$$\alpha_{\lim}=\lim_{i\to\infty}\frac{\tilde{\phi}(\hat{i})}{-\psi(\hat{i})}=\lim_{i\to\infty}\frac{\tilde{\phi}(\hat{i})}{-\tilde{\psi}(\hat{i})}=\lim_{i\to\infty}\frac{\tilde{\phi}(\bar{i})}{-\tilde{\psi}(\bar{i})}=\lim_{i\to\infty}\frac{\log p_i}{\log s_i}.$$

Now, we construct examples for the case: $0 < \alpha_{\lim} < \infty$.

3.12 The Case $0 < \alpha_{\lim} < \infty$

Our examples show that the multifractal spectrum has zero to three phase transitions when $0 < \alpha_{\lim} < \infty$.

3.12.1 Example of Zero Phase Transitions

We construct an example in which the multifractal spectrum is analytic everywhere. First, we construct the potentials for our example. For each $i \in \mathbb{N}$, take $0 < p_i = \frac{C}{(i+1)^3} < 1$ such that C satisfies

$$\sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{C}{(i+1)^3} = 1. \tag{3.12.1}$$

Consider an arbitrary $x = (x_1, x_2, x_3, \ldots) \in \Sigma$. Respectively, define the locally constant potentials $\tilde{\phi} : \Sigma \to \mathbb{R}^-$ and $\tilde{\psi} : \Sigma \to \mathbb{R}^+$ as follows:

$$\tilde{\phi}(x) = \log p_{x_1} = \log\left(\frac{C}{(x_1+1)^3}\right) \text{ and } \tilde{\psi}(x) = \log\left(\frac{6}{\pi^2 (x_1)^2}\right).$$

Now, we prove that our chosen potentials satisfy the conditions of Theorem 3.1.5. We find that

$$\mathscr{P}(-\tilde{\psi}) = \log\left(\sum_{i=1}^{\infty} \frac{6}{\pi^2 (i)^2}\right) = 0.$$

Because of Inequality (3.11.2) and ψ is a metric potential, $\tilde{\psi}$ is also a metric potential. We find that

$$\mathscr{P}(\tilde{\phi}) = \log\left(\sum_{i=1}^{\infty} \frac{C}{(i+1)^3}\right) = 0$$

by Equation (3.12.1). Define a Bernoulli measure μ such that $\mu([i]) = p_i$ for every $i \in \mathbb{N}$ and observe that

$$\mu(\Sigma) = \sum_{i=1}^{\infty} \mu([i]) = \sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{C}{(i+1)^3} = 1.$$

Furthermore,

$$\mu([x_1, \ldots, x_m]) = p_{x_1} \cdots p_{x_m} = \exp\left(\sum_{i=1}^{m} \log p_{x_i}\right) = \exp\left(\sum_{i=0}^{m-1} \tilde{\phi}(\sigma^i(x))\right),$$

for any cylinder $[x_1, \ldots, x_m] \subset \Sigma$, so μ is the Gibbs measure for $\tilde{\phi}$. For any locally Hölder $u : \Sigma \to \mathbb{R}$,

$$\tilde{\phi}(x) - (-\tilde{\psi}(x)) = \log\left(\frac{C}{(x_1+1)^3}\right) - \log\left(\frac{6}{\pi^2 (x_1)^2}\right) \neq 0 = u(x) - u(x) = u(x) - u \circ \sigma(x)$$

for the sequence $x = (2, 2, 2, \ldots)$. Hence, $\tilde{\phi}$ is non-cohomologous to $\tilde{\psi}$.

Take any $q, t \in \mathbb{R}$. Consider the potential $q\tilde{\phi} - t\tilde{\psi}$. The set $Q = \{q \in \mathbb{R} : T(q) > \tilde{t}(q)\}$ is correlated to the number of possible phase transitions for the multifractal spectrum. We will show that $Q = \emptyset$ because the multifractal spectrum has no phase transitions in that case (see Proposition 3.7.3). For this analysis, we must consider the functions $\tilde{t}(q)$ and $T(q)$. By Proposition 3.3.6, $\tilde{t}(q) = -\alpha_{\lim} q + t_\infty$. Consider $\bar{j} = (j, j, ...)$, for each $j \in \mathbb{N}$. We calculate

$$\alpha_{\lim} = \lim_{j\to\infty} \frac{\tilde{\phi}(\bar{j})}{-\tilde{\psi}(\bar{j})} = \lim_{j\to\infty} \frac{-3\log j}{-2\log j} = \frac{3}{2}. \tag{3.12.2}$$

We use Lemma 3.3.6 to find the value of t_∞.

Remember that

$$t_\infty = \inf\{t \in \mathbb{R} : \mathscr{P}(-t\tilde{\psi}) < \infty\} = \inf\{t \in \mathbb{R} : Z_1(-t\tilde{\psi}) < \infty\}.$$

Because

$$\begin{aligned} Z_1(-t\tilde{\psi}) &= \sum_{x_1=1}^{\infty} \exp \sup_{x\in[x_1]} \left(-2t\log x_1 - t\log\left(\frac{6}{\pi^2}\right)\right) \\ &= \sum_{j=1}^{\infty} \exp\left(\log(j^{-2t}) - t\log\left(\frac{6}{\pi^2}\right)\right) < \infty \end{aligned} \tag{3.12.3}$$

for $t > \frac{1}{2}$, we find that $t_\infty = \frac{1}{2}$. Thus,

$$\tilde{t}(q) = -\frac{3}{2}q + \frac{1}{2}$$

by Equations (3.12.2) and (3.12.3).

We find that

$$Q = \{q \in \mathbb{R} : T(q) = \tilde{t}(q)\} = \{q \in \mathbb{R} : \mathscr{P}(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) \le 0\} = \{q \in \mathbb{R} : Z_1(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) \le 1\}$$

because $\mathscr{P}(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) = \log(Z_1(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}))$ by Equation (3.11.3).

Consider $\bar{j} = (j, j, j....)$ for each $j \in \mathbb{N}$. For each $q \in \mathbb{R}$,

$$\begin{aligned} Z_1(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) &= \sum_{x_1=1}^{\infty} \exp \sup_{x\in[x_1]} (q\tilde{\phi} - \tilde{t}(q)\tilde{\psi})(x) = \sum_{j=1}^{\infty} \exp((q\tilde{\phi} - \tilde{t}(q)\tilde{\psi})(\bar{j})) \\ &= \sum_{j=1}^{\infty} \frac{(C)^q \left(\frac{6}{\pi^2}\right)^{-\frac{3}{2}q+\frac{1}{2}}}{j^{3q}(j+1)^{2\tilde{t}(q)}} = \infty, \end{aligned}$$

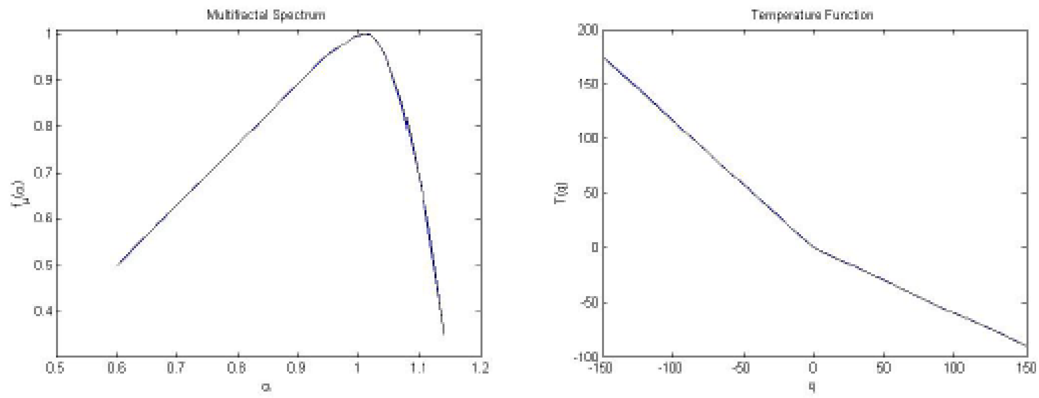
so $T(q) \ne \tilde{t}(q)$. Hence,

$$Q = \{q \in \mathbb{R} : Z_1(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) \le 1\} = \{q \in \mathbb{R} : \mathscr{P}(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) \le 0\} = \emptyset.$$

Therefore, the multifractal spectrum has no phase transitions by Proposition 3.7.3.

3.12.2 One Phase Transition

We construct an example in which the multifractal spectrum has one phase transition. A picture of this example is as follows.



(A) Multifractal Spectrum on $(0.6, 1.13653)$ (B) Temperature Function on $(-150, 150)$

FIGURE 3.1. We observe that $\alpha_{\lim} = 0.6$ and $T(q)$ has phase transition around $q = 1.3$. Most importantly, the multifractal spectrum has its phase transition around $\alpha \in (0.955, 0.999)$ and its maximum is $f_\mu(\alpha(0)) = 1$ (such that $\alpha(0) = 1.0068$).

First, we construct the potentials for our example. For each $i \in \mathbb{N}$, let $0 < p_i = \frac{C}{(i)^{\frac{6}{5}}(\log(i+2))^2} < 1$ such that C satisfies

$$\sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{C}{(i)^{\frac{6}{5}}(\log(i+2))^2} = 1. \tag{3.12.4}$$

Consider an arbitrary $x = (x_1, x_2, x_3, \ldots) \in \Sigma$. Respectively, define the locally constant potentials $\tilde{\phi} : \Sigma \to \mathbb{R}^-$ and $\tilde{\psi} : \Sigma \to \mathbb{R}^+$ as follows:

$$\tilde{\phi}(x) = \log p_{x_1} = \log\left(\frac{C}{(x_1)^{\frac{6}{5}}(\log(x_1+2))^2}\right) \text{ and } \tilde{\psi}(x) = -\log\left(\frac{6}{\pi^2 (x_1)^2}\right).$$

Now, we prove that our chosen potentials satisfy the conditions of Theorem 3.1.5. We find that

$$\mathscr{P}(-\tilde{\psi}) = \log\left(\sum_{i=1}^{\infty} \frac{6}{\pi^2 (i)^2}\right) = 0.$$

Because of Inequality (3.11.2) and ψ is a metric potential, $\tilde{\psi}$ is also a metric potential. We find that

$$\mathscr{P}(\tilde{\phi}) = \log\left(\sum_{i=1}^{\infty} \frac{C}{(i)^{\frac{6}{5}}(\log(i+2))^2}\right) = 0$$

by Equation (3.12.4). Define a Bernoulli measure μ such that $\mu([i]) = p_i$ for every $i \in \mathbb{N}$ and observe that

$$\mu(\Sigma) = \sum_{i=1}^{\infty} \mu([i]) = \sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{C}{(x_1)^{\frac{6}{5}}(\log(i+2))^2} = 1.$$

Furthermore,

$$\mu([x_1,...,x_m]) = p_{x_1}\cdots p_{x_m} = \exp\left(\sum_{i=1}^{m}\log p_{x_i}\right) = \exp\left(\sum_{i=0}^{m-1}\tilde{\phi}(\sigma^i(x))\right),$$

for any cylinder $[x_1,...,x_m] \subset \Sigma$, so μ is the Gibbs measure for $\tilde{\phi}$. For any locally Hölder $u : \Sigma \to \mathbb{R}$,

$$\tilde{\phi}(x) - (-\tilde{\psi}(x)) = \log\left(\frac{C}{(x_1)^{\frac{6}{5}}(\log(x_1+2))^2}\right) - \log\left(\frac{6}{\pi^2(x_1)^2}\right) \neq 0 = u(x) - u(x) = u(x) - u\circ\sigma(x)$$

such that $x = (2,2,2,...)$. Hence, $\tilde{\phi}$ is non-cohomologous to $\tilde{\psi}$.

Take any $q,t \in \mathbb{R}$. Consider the potential $q\tilde{\phi} - t\tilde{\psi}$. We will show that there exists a $q_0 > \frac{6}{5}$, such that $Q = [q_0,\infty)$, because the multifractal spectrum has one phase transition in that case. For this analysis, we must consider the functions $\tilde{t}(q)$ and $T(q)$. By Proposition 3.3.6, $\tilde{t}(q) = -\alpha_{\lim} q + t_\infty$. Consider $\bar{j} = (j,j,...)$ for each $j \in \mathbb{N}$. We calculate

$$\alpha_{\lim} = \lim_{j\to\infty}\frac{\tilde{\phi}(\bar{j})}{-\tilde{\psi}(\bar{j})} = \lim_{j\to\infty}\frac{-\frac{6}{5}\log j}{-2\log j} = \frac{3}{5}.$$

Again, $Z_1(-t\tilde{\psi}) < \infty$ for $t > \frac{1}{2}$, so $t_\infty = \frac{1}{2}$. Hence, we find that

$$\tilde{t}(q) = -\frac{3}{5}q + \frac{1}{2}.$$

To prove that there exists a $q_0 > \frac{6}{5}$, such that $Q = [q_0,\infty)$, we take the following steps. We will prove that $\mathscr{P}(q\phi - \tilde{t}(q)\psi) > 0$ for a fixed $q < \frac{6}{5}$ and $\mathscr{P}(q\phi - \tilde{t}(q)\psi) \le 0$ for a fixed $q > \frac{6}{5}$. To prove this, it is sufficient to prove that there exists a $q > \frac{6}{5}$ such that $Z_1(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) \le 1$ by Equation (3.11.3). Fix any $q \in \mathbb{R}$. We get that

$$\begin{aligned} Z_1(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) &= \sum_{x_1=1}^{\infty}\exp\sup_{x\in[x_1]}(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi})(x) = \sum_{j=1}^{\infty}\exp(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi})(\bar{j})) \\ &= \sum_{j=1}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}. \end{aligned} \tag{3.12.5}$$

For each sufficiently small $\delta > 0$, there exists an $N \in \mathbb{N}$ such that

$$\left|\sum_{j=1}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}} - \sum_{j=1}^{N}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}\right| \le \delta. \tag{3.12.6}$$

Take a sufficiently small δ and an $N \in \mathbb{N}$ that satisfies Inequality (3.12.6). Then,

$$\sum_{j=1}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}} = \sum_{j=1}^{N-1}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}} + \sum_{j=N}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}.$$

Let

$$f(q)=\sum_{j=N}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}=\sum_{j=N}^{\infty}\frac{((C)(\frac{6}{\pi^2})^{-\frac{3}{5}})^q(\frac{6}{\pi^2})^{\frac{1}{2}}}{(j)(\log(j+2))^{2q}}.$$

For each sufficiently small $\varepsilon>0$, there exists a $\widehat{q}>\frac{6}{5}$ such that

$$f'(\widehat{q})=\sum_{j=N}^{\infty}\frac{\left((C)\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}}\right)^{\widehat{q}}\left(\frac{6}{\pi^2}\right)^{\frac{1}{2}}\log\left((C)\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}}\right)}{(j)(\log(j+2))^{2\widehat{q}}}-2\sum_{j=N}^{\infty}\frac{\left((C)\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}}\right)^{\widehat{q}}\left(\frac{6}{\pi^2}\right)^{\frac{1}{2}}}{j\log\log(j+2)(\log(j+2))^{2\widehat{q}}}<\varepsilon.$$

Let $g(q)=\sum_{j=1}^{N-1}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}$. Similar calculations give us that $g'(\widehat{q})<\varepsilon$ for a $\widehat{q}>\frac{6}{5}$. Hence,

$$f(q)=\sum_{j=N}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}\text{ and }g(q)=\sum_{j=1}^{N-1}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}$$

decrease with respect to q. Then,

$$\sum_{j=1}^{N-1}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}\le\int_{1}^{N-1}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}\,\mathrm{d}\mu_q\le 1-\varepsilon \tag{3.12.7}$$

$$\text{and }\sum_{j=N}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}\le\int_{N}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}\,\mathrm{d}\mu_q\le\varepsilon. \tag{3.12.8}$$

for a $q>\frac{6}{5}$. Therefore, by Equations (3.12.7) and (3.12.8),

$$Z_1(q\tilde{\phi}-\tilde{t}(q)\tilde{\psi})<\sum_{j=1}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}\le 1 \tag{3.12.9}$$

for a fixed $q>\frac{6}{5}$. Hence, $\mathscr{P}(q\phi-\tilde{t}(q)\psi)\le 0$ for a fixed $q>\frac{6}{5}$.

Now, we will prove that there exists a value $1<q<\frac{6}{5}$ such that

$$\mathscr{P}(q\tilde{\phi}-\tilde{t}(q)\tilde{\psi})>0.$$

We find that

$$\mathscr{P}(q\tilde{\phi}-\tilde{t}(q)\tilde{\psi})=\sum_{j=1}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}$$

by Equation (3.11.3). We find that

$$\sum_{j=1}^{\infty}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}>\sum_{j=1}^{25}\frac{(C)^q\left(\frac{6}{\pi^2}\right)^{-\frac{3}{5}q+\frac{1}{2}}}{(j)(\log(j+2))^{2q}}>1 \tag{3.12.10}$$

if $q = 1.15$. Hence, there exists a value $1 < q < \frac{6}{5}$ (namely, $q = 1.15$) such that

$$\mathscr{P}(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) > 0.$$

By Inequality (3.12.9), there exists a $q > \frac{6}{5}$ such that $T(q) = \tilde{t}(q)$ and by Inequality (3.12.10), there exists a $q < \frac{6}{5}$ such that $T(q) > \tilde{t}(q)$. Thus, there exists a $q_0 > \frac{6}{5}$ such that

$$Q = \{q \in \mathbb{R} : T(q) = \tilde{t}(q)\} = \{q \in \mathbb{R} : \mathscr{P}(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) < 0\} = [q_0, \infty).$$

To prove that the multifractal spectrum has one phase transition, we need to prove that $\alpha^- > \alpha_{\lim}$. We note that the measure μ_q, defined on each $[x_1, \ldots, x_m] \subset \Sigma$ and every $x \in [x_1, \ldots, x_m]$ by

$$\mu_q([x_1, \ldots, x_m]) = \exp\left(\sum_{i=0}^{m-1} (q\tilde{\phi} - T(q)\tilde{\psi})(\sigma^i(x))\right),$$

is a Gibbs measure for $q\tilde{\phi} - T(q)\tilde{\psi}$ for each $q \in Q^{\complement}$. Now, we consider α^-. We get that

$$\begin{aligned} \int -\tilde{\phi}\, \mathrm{d}\mu_q &= \sum_{j=1}^{\infty} -\frac{(C)^q(\frac{6}{\pi^2})^{T(q)}}{j^{\frac{6}{5}q+2T(q)}(\log(j+2))^{2q}} \log\left(\frac{C}{j^{\frac{6}{5}}(\log(j+2))^2}\right) \\ &< \sum_{j=1}^{\infty} \frac{1}{j^{\frac{6}{5}q+2T(q)-1}(\log(j+2))^{2q-1}} < \infty \end{aligned} \tag{3.12.11}$$

and

$$\begin{aligned} \int \tilde{\psi}\, \mathrm{d}\mu_q &= -\sum_{j=1}^{\infty} \frac{(C)^q(\frac{6}{\pi^2})^{T(q)}}{j^{\frac{6}{5}q+2T(q)}(\log(j+2))^{2q}} \log\left(\frac{6}{\pi^2 j^2}\right) \\ &< \sum_{j=1}^{\infty} \frac{1}{j^{\frac{6}{5}q+2T(q)-1}(\log(j+2))^{2q-1}} < \infty \end{aligned} \tag{3.12.12}$$

for $q_0 > q > \frac{6}{5}$. We remark that both integrals are infinite if $q < \frac{5}{6}$. Hence,

$$\alpha^- = \lim_{q \to q_0^-} \frac{\int \phi\, \mathrm{d}\mu_q}{\int -\psi\, \mathrm{d}\mu_q} > \alpha_{\lim} \tag{3.12.13}$$

for a $q_0 > \frac{6}{5}$ by Inequalities (3.12.11) and (3.12.12).

Using the techniques from the proof of Theorem 3.1.5, we find the following. Given that $q_0 > \frac{6}{5}$, $\alpha^- > \alpha_{\lim} = \alpha_{\inf}$ by Equation (3.12.13). Then, the multifractal spectrum f_μ is analytic on $(\alpha_{\inf}, \alpha^-)$ and $(\alpha^-, \alpha_{\sup})$. It equals $T(q_0) + q_0\alpha$ on $(\alpha_{\inf}, \alpha^-)$, is concave on $(\alpha^-, \alpha_{\sup})$, and has its maximum at $\alpha(0)$. The multifractal spectrum only has one phase transition at α^-.

3.12.3 Two and Three Phase Transitions

We will form an example in which the multifractal spectrum has two or three phase transitions. First, we construct the potentials for our example. Choose a $k \in \mathbb{N}$. Let

$$p_k = \frac{C_k}{(k)^{\frac{6}{5}}(\log(k+2))^2}$$

and for each $i \in \mathbb{N} \setminus \{k\}$, take
$$p_i = \frac{C}{(i)^{\frac{6}{5}}(\log(i+2))^2}$$
such that $0 < C < 1$ and $C_k > 1$ satisfy $\sum_{i=1}^{\infty} p_i = 1$.

Consider an arbitrary $x = (x_1, x_2, x_3, \ldots) \in \Sigma$. Respectively, define the potentials $\phi : \Sigma \to \mathbb{R}^-$ and $\tilde{\psi} : \Sigma \to \mathbb{R}^+$ as follows:
$$\phi(x) = \log p_{x_1} = \log\left(\frac{C}{(i)^{\frac{6}{5}}(\log(i+2))^2}\right)$$
if $x_1 = i \neq k$,
$$\phi(x) = \log p_{x_1} = \log\left(\frac{C_k}{(k)^{\frac{6}{5}}(\log(k+2))^2}\right)$$
if $x_1 = k$, and
$$\tilde{\psi}(x) = -\log\left(\frac{6}{\pi^2(x_1)^2}\right).$$
We will take $k \neq 2$ for our calculations.

Now, we will prove that $\tilde{\phi}$ and $\tilde{\psi}$ satisfy the conditions for Theorem 3.1.5. We find that
$$\mathscr{P}(-\tilde{\psi}) = \log\left(\sum_{i=1}^{\infty} \frac{6}{\pi^2(i)^2}\right) = 0.$$
Because of Inequality (3.11.2) and ψ is a metric potential, $\tilde{\psi}$ is also a metric potential. Because $\sum_{i=1}^{\infty} p_i = 1$,
$$\mathscr{P}(\tilde{\phi}) = \log\left(\sum_{i=1}^{\infty} p_i\right) = 0.$$

Define a Bernoulli measure μ such that $\mu([i]) = p_i$ for every $i \in \mathbb{N}$ and observe that $\mu(\Sigma) = \sum_{i=1}^{\infty} \mu([i]) = \sum_{i=1}^{\infty} p_i = 1$. Furthermore,
$$\mu([x_1, \ldots, x_m]) = p_{x_1} \cdots p_{x_m} = \exp\left(\sum_{i=1}^{m} \log p_{x_i}\right) = \exp\left(\sum_{i=0}^{m-1} \tilde{\phi}(\sigma^i(x))\right),$$
for any cylinder $[x_1, \ldots, x_m] \subset \Sigma$, so μ is the Gibbs measure for $\tilde{\phi}$. For any locally Hölder $u : \Sigma \to \mathbb{R}$,
$$\tilde{\phi}(x) - (-\tilde{\psi}(x)) = \log\left(\frac{C}{(x_1)^{\frac{6}{5}}(\log(x_1+2))^2}\right) - \log\left(\frac{6}{\pi^2(x_1)^2}\right) \neq 0 = u(x) - u(x) = u(x) - u \circ \sigma(x)$$
such that $x = (2, 2, 2, \ldots)$. Hence, $\tilde{\phi}$ is non-cohomologous to $\tilde{\psi}$.

Consider $\bar{j} := (j, j, j, \ldots)$ for each $j \in \mathbb{N}$.
$$\alpha_{\lim} = \lim_{j \to \infty} \frac{\tilde{\phi}(\bar{j})}{-\tilde{\psi}(\bar{j})} = \lim_{j \to \infty} \frac{-\frac{6}{5}\log j}{-2\log j} = \frac{3}{5}.$$
Again, we find that $Z_1(-t\psi) < \infty$ for $t > \frac{1}{2}$, so $t_\infty = \frac{1}{2}$. Hence,
$$\tilde{t}(q) = -\frac{3}{5}q + \frac{1}{2}.$$

We prove that the following is possible: there exist $q_1 > q_0 > \frac{5}{6}$ such that $Q = [q_0, q_1]$ because the multifractal spectrum will have either two or three phase transitions in this case.

Because the argument is nearly identical to the previous example, we instead give an outline of it. We would prove that

$$\mathscr{P}(q\phi - \tilde{t}(q)\psi) > 0 \tag{3.12.14}$$

for a fixed $q < \frac{5}{6}$, $Z_1(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) \le 1$ for a $q_0 > q > \frac{5}{6}$, and $\mathscr{P}(q\phi - \tilde{t}(q)\psi) > 0$ for a fixed $\frac{5}{6} < q < q_1 < \infty$. If $Q = [q_0, q_1]$ such that $q_0 < \frac{5}{6} < q_1$, we would only need to change one step of the preceding procedure: instead of proving Inequality (3.12.14), we would need to prove that $\mathscr{P}(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) \le 0$ for a fixed $q < \frac{5}{6}$.

Consider $\bar{j} = (j, j, j, \ldots)$ for each $j \in \mathbb{N}$. We would need

$$\frac{\log p_k}{-2\log(k)} > \alpha_{\lim} \tag{3.12.15}$$

so that $T(q)$ can have a second phase transition at q_1 (hence, there would exist a $q > q_1$ such that $\alpha(q) > \alpha_{\lim}$). Recall that $\alpha_{\lim} = \lim_{j\to\infty} \frac{\log p_j}{\log|G'(\pi(\bar{j}))|} = \lim_{j\to\infty} \frac{\log p_j}{2\log(j)}$. Hence, for each sufficiently small $\varepsilon > 0$, there exist $j \ge J$ such that

$$\alpha_{\lim} - \varepsilon \le \frac{\log p_j}{-2\log j} \le \alpha_{\lim} + \varepsilon. \tag{3.12.16}$$

Choose a sufficiently small $\varepsilon > 0$ and take a $j \ge k$ that satisfies Inequality (3.12.16). Thus, we must satisfy

$$\frac{\log p_k}{-2\log(k)} > \frac{\log p_j}{-2\log j}. \tag{3.12.17}$$

Because we assumed that $C_k > 1$, the p_k satisfies Inequality (3.12.17) and hence, Inequality (3.12.15). Thus, $T(q)$ has a phase transition at q_1.

We note that the measure μ_q, defined on each $[x_1, \ldots, x_m] \subset \Sigma$ and every $x \in [x_1, \ldots, x_m]$ by

$$\mu_q([x_1, \ldots, x_m]) = \exp\left(\sum_{i=0}^{m-1} (q\tilde{\phi} - T(q)\tilde{\psi})(\sigma^i(x))\right),$$

is a Gibbs measure for $q\tilde{\phi} - T(q)\tilde{\psi}$ for each $q \in Q^{\complement}$. Now, we must determine whether α^+ and α^- are finite. The computations to show that

$$\int -\tilde{\phi}\, \mathrm{d}\mu_q < \infty \text{ and} \int \tilde{\psi}\, \mathrm{d}\mu_q < \infty \tag{3.12.18}$$

for each $q > \frac{5}{6}$ are identical to Inequalities (3.12.11) and (3.12.12). Furthermore, for each $q < \frac{5}{6}$,

$$\int -\tilde{\phi}\, \mathrm{d}\mu_q = \infty \text{ and} \int \tilde{\psi}\, \mathrm{d}\mu_q = \infty \tag{3.12.19}$$

as found in Subsection 3.12.2.

Therefore, using the techniques from the proof of Theorem 3.1.5, the multifractal spectrum has two or three phase transitions depending on the values of q_0 and q_1 :

1. If $q_0 \le \frac{5}{6} < q_1$, then $\alpha^- = \alpha_{\lim} > \alpha^+$ by Inequalities (3.12.18) and (3.12.19). The multifractal spectrum is analytic and concave on $(\alpha_{\inf}, \alpha^+)$, (α^+, α^-), and $(\alpha^-, \alpha_{\sup})$. Furthermore, $f_\mu(\alpha) = T(q_1) + q_1\alpha$ on (α^+, α^-) and has its maximum at $\alpha(0)$. The multifractal spectrum has phase transitions at $\alpha_{\lim}$ and α^+.

2. If $\frac{5}{6} < q_0 < q_1$, then $\alpha^- > \alpha_{\lim} > \alpha^+$ by Inequality (3.12.18). The multifractal spectrum is analytic and concave on $(\alpha_{\inf}, \alpha^+)$, $(\alpha^+, \alpha_{\lim})$, $(\alpha_{\lim}, \alpha^-)$, and $(\alpha^-, \alpha_{\sup})$. Furthermore,

$$f_\mu(\alpha) = \begin{cases} T(q_1) + q_1\alpha & \text{on } (\alpha^+, \alpha_{\lim}) \\ T(q_0) + q_0\alpha & \text{on } (\alpha_{\lim}, \alpha^-) \end{cases}$$

and the multifractal spectrum has its maximum at $\alpha(0)$. Then, it has phase transitions at $\alpha^-, \alpha_{\lim}$, and α^+.

3.13 The Case $\alpha_{\lim} = \infty$

By Theorem 3.1.6, the multifractal spectrum has zero to one phase transition if $\alpha_{\lim} = \infty$. We construct an example in which the multifractal spectrum has one phase transition.

Let $\Sigma = \mathbb{N}^{\mathbb{N}}$. First, we define our potentials $\phi : \Sigma \to \mathbb{R}^-$ and $\psi : \Sigma \to \mathbb{R}^+$ as follows. For each $i \in \mathbb{N}$, let

$$p_i = C\left(\frac{1}{2}\right)^{\sqrt{i}}$$

such that C satisfies $\sum_{i=1}^\infty p_i = 1$. We remark that p_i is similar to the Minkowski ?−function. We will respectively take the potentials $\tilde{\psi}$ with $\tilde{\psi}(x) = -\log\left(\frac{6}{\pi^2 x_1^2}\right)$ and we let

$$\phi(x) = \log p_{x_1} = \log\left(C\left(\frac{1}{2}\right)^{\sqrt{x_1}}\right).$$

for each $x \in \Sigma$. Denote ϕ as $\tilde{\phi}$.

Now, we prove that ϕ and ψ satisfy the conditions of Theorem 3.1.6. We find that

$$\mathscr{P}(-\tilde{\psi}) = \log\left(\sum_{i=1}^\infty \frac{6}{\pi^2 (i)^2}\right) = 0.$$

Because of Inequality (3.11.2) and ψ is a metric potential, $\tilde{\psi}$ is also a metric potential. Because $\sum_{i=1}^\infty p_i = 1$,

$$\mathscr{P}(\tilde{\phi}) = \log\left(\sum_{i=1}^\infty p_i\right) = 0.$$

Define a Bernoulli measure μ such that $\mu([i]) = p_i$ for every $i \in \mathbb{N}$ and observe that

$$\mu(\Sigma) = \sum_{i=1}^\infty \mu([i]) = \sum_{i=1}^\infty p_i = 1.$$

Furthermore,

$$\mu([x_1,...,x_m]) = p_{x_1}\cdots p_{x_m} = \exp\left(\sum_{i=1}^{m}\log p_{x_i}\right) = \exp\left(\sum_{i=0}^{m-1}\tilde{\phi}(\sigma^i(x))\right),$$

for any cylinder $[x_1,...,x_m]\subset\Sigma$, so μ is the Gibbs measure for $\tilde{\phi}$. For any locally Hölder $u:\Sigma\to\mathbb{R}$,

$$\tilde{\phi}(x)-(-\tilde{\psi}(x)) = -\sqrt{x_1}\log(2)+\log C-\log\left(\frac{6}{\pi^2(x_1)^2}\right)\neq 0 = u(x)-u(x) = u(x)-u\circ\sigma(x)$$

such that $x=(1,1,1,...)$. Hence, $\tilde{\phi}$ is non-cohomologous to $\tilde{\psi}$.

Now, we will try to find an expression for Q because it will help us prove that the multifractal spectrum has one phase transition. Consider $\bar{j}=(j,j,j,...)$ for each $j\in\mathbb{N}$. Because $\tilde{\phi}$ and $\tilde{\psi}$ are locally constant,

$$\mathscr{P}(q\tilde{\phi}-t\tilde{\psi}) = \log Z_1(q\tilde{\phi}-t\tilde{\psi}) = \log\left(\sum_{j=1}^{\infty}\frac{C^q(\frac{6}{\pi^2})^t}{2^{\sqrt{j}q}(j)^{2t}}\right)$$

by Equation (3.11.3). We analyse $\mathscr{P}(q\tilde{\phi}-t\tilde{\psi})$ as follows.

1. For each $q<0$,

   $$\log\left(\sum_{j=1}^{\infty}\frac{C^q(\frac{6}{\pi^2})^t}{2^{\sqrt{j}q}(j)^{2t}}\right)=\infty \tag{3.13.1}$$

   independent of the choice of $t\in\mathbb{R}$. Then, $T(q)=\tilde{t}(q)=\infty$ for these q.

2. If $q=0$,

   $$\log\left(\sum_{j=1}^{\infty}\frac{(\frac{6}{\pi^2})^t}{(j)^{2t}}\right)<\infty \tag{3.13.2}$$

   for every $t>\frac{1}{2}\in\mathbb{R}$. Hence, $\tilde{t}(0)=\frac{1}{2}<T(0)$. In fact, $T(0)=1$.

3. For each $q>0$,

   $$\log\left(\sum_{j=1}^{\infty}\frac{C^q(\frac{6}{\pi^2})^t}{2^{\sqrt{j}q}(j)^{2t}}\right)<\infty \tag{3.13.3}$$

   independent of the choice of $t\in\mathbb{R}$. Then, $-\infty=\tilde{t}(q)<T(q)$ for these q.

By Lemma 3.3.4 and Inequalities (3.13.1)-(3.13.3),

$$\tilde{t}(q)=\begin{cases}\infty & \text{if } q<0\\ \frac{1}{2} & \text{if } q=0\\ -\infty & \text{if } q>0.\end{cases}$$

By Inequalities (3.13.1)-(3.13.3),

$$Q=\{q\in\mathbb{R}:T(q)=\tilde{t}(q)\}=(-\infty,0).$$

Denote

$$\alpha(0) := \lim_{q\to 0^+} \alpha(q).$$

We find that $\alpha_{\lim} = \lim_{i\to\infty} \frac{\log p_i}{\log s_i} = \infty$.

We will prove that $\alpha(0) < \alpha_{\lim} = \infty$. We note that the measure μ_q, defined on each $[x_1,...,x_m] \subset \Sigma$ and every $x \in [x_1,...,x_m]$ by

$$\mu_q([x_1,...,x_m]) = \exp\left(\sum_{i=0}^{m-1}(q\tilde{\phi} - T(q)\tilde{\psi})(\sigma^i(x))\right),$$

is a Gibbs measure for $q\tilde{\phi} - T(q)\tilde{\psi}$ for each $q \in Q^{\complement}$. Because $T(0) = 1 > \tilde{t}(0)$,

$$\lim_{q\to 0^+} \int -\phi \, d\mu_q \le \lim_{q\to 0^+} \sum_{j=1}^{\infty} \frac{C^q(\frac{6}{\pi^2})^{T(q)}[\sqrt{j}\log(2) + \log C]}{2^{\sqrt{j}q}(j)^{2T(q)}} \in (-\infty,\infty)$$

$$\text{and } \lim_{q\to 0^+} \int \psi \, d\mu_q \le \lim_{q\to 0^+} \sum_{j=1}^{\infty} \frac{C^q(\frac{6}{\pi^2})^{T(q)}[2\log(j) - \log(\frac{6}{\pi^2})]}{2^{\sqrt{j}q}(j)^{2T(q)}} \in (-\infty,\infty).$$

Hence, $\alpha(0) < \alpha_{\lim} = \infty = \alpha_{\sup}$. We formalise our analysis of the behaviour of the multifractal spectrum as follows.

Proposition 3.13.1. *In this example, the multifractal spectrum is increasing and analytic on* $(\alpha_{\inf}, \alpha(0))$ *and* $f_\mu(\alpha) = T(0)$ *on* $(\alpha(0),\infty)$. *Furthermore, the multifractal spectrum has a phase transition at* $\alpha(0)$.

Proof. For each $\alpha \in (\alpha_{\inf}, \alpha(0))$, there exists a unique $q > 0$ such that $\alpha = \alpha(q)$. Because each $q \in Q^{\complement}$, $f_\mu(\alpha)$ is analytic on $(\alpha_{\inf}, \alpha(0))$. The increasing behaviour of the multifractal spectrum on $(\alpha_{\inf}, \alpha(0))$ follows from Proposition 3.6.4. For each $\alpha \in (\alpha(0),\infty)$, $f_\mu(\alpha) = \inf_{q\in\mathbb{R}}\{T(q) + q\alpha\} = T(0)$. ∎

3.14 The Case When $\alpha_{\lim}$ Does Not Exist

When $\alpha_{\lim}$ does not exist, the multifractal spectrum can have infinitely many phase transitions. We remark that Iommi and Jordan [IJ13] create a similar example in the setting of a suspension flow.

Now, we roughly outline the procedure for creating such an example in our setting. First, we take the locally constant potentials ϕ and ψ such that

$$\phi(x) = \log p_{x_1} \text{ and } \psi(x) = \log s_{x_1}^{-1}$$

for each $x = (x_1, x_2, ...) \in \Sigma$. Then, to define the p_i for each $i \in \mathbb{N}$, we partition the natural numbers as follows. Let $r_0 = 0$ and $r_1 = 1$. Consider the infinite sequence $\{r_k\}_{k=2}^{\infty}$ of primes $\{2,3,5,7,11,...\}$. We define the sets $\{I_k\}_{k\in\mathbb{N}_0}$ as follows:

$$I_0 := \{m \in \mathbb{N} \text{ such that } m \text{ cannot be written as any prime power of any } n \in \mathbb{N}\}.$$

$$I_1 := \{m \in \mathbb{N} \text{ that can be written as the 2nd power of some } n \in \mathbb{N}\}$$

In general,

$$I_k := \{m \in \mathbb{N} \text{ that can be written as the } r_{k+1}\text{st power of some } n \in \mathbb{N}\}.$$

We will now define p_{x_1} and s_{x_1} for our potentials. First, respectively define increasing sequences of constants $\{C_k\}_{k\in\mathbb{N}_0}$ and $\{M_k\}_{k\in\mathbb{N}_0}$ as follows. The terms of both sequences are chosen such that $\sum_{m=1}^{\infty} p_m = 1$. For each $k \in \mathbb{N}\cup\{0\}$, we have a recursive relation for the values of the sequence $\{l_k\}$ in the following expression for p_m. For each $m \in \mathbb{N}$, we get that

$$p_m = \frac{C_k}{m^{l_k}(\log(m+2))^{M_k}}$$

if $k \in \mathbb{N}_0$ and $m \in I_k$. We remind the reader that locally Hölder potentials can be approximated by locally constant potentials. For each $m \in \mathbb{N}$, define

$$s_m := \frac{6}{\pi^2 m^2}$$

for each $m \in \mathbb{N}$.

For each I_k, there exists a function $\tilde{t}_k(q)$ as follows:

$$\tilde{t}_k(q) := \inf\{t \in \mathbb{R} : \sum_{I_k} p_m^q s_m^{\tilde{t}_k(q)} < \infty\}.$$

We define

$$\tilde{t}(q) = \left\{ \sup_{q\in\mathbb{R}} \tilde{t}_k(q) : k \in \mathbb{N}\cup\{0\} \right\}.$$

By construction, $|\tilde{t}'_k| > |\tilde{t}'_{k+1}|$ and each $\tilde{t}_k$ is linear. We find that the phase transitions for $\tilde{t}(q)$ occur at values of q such that $\tilde{t}_k(q) = \tilde{t}_{k+1}(q)$. These occur at each $q \in \mathbb{N}$; hence, $\tilde{t}(q)$ has infinitely many phase transitions.

Finally, we prove that $Z_1(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) < 1$ for each $q \in \bigcup_{k\in\mathbb{N}} (k, k+1) \cup (a, 1)$ for some $a \in \mathbb{R}$. This gives us that $\mathscr{P}(q\tilde{\phi} - \tilde{t}(q)\tilde{\psi}) < 1$ for those q; hence, $T(q) = \tilde{t}(q)$ for each $q \in \mathbb{R}^+ \setminus N \cup (0,1)$ (for some $1 > \bar{q} \geq a$). Using techniques from the previous example, we get results for a possible phase transition for the multifractal spectrum at $\alpha(\bar{q})$. Without loss of generality, let us assume that there is no phase transition at $\alpha(\bar{q})$.

Hence, the multifractal spectrum behaves as follows. $f_\mu(\alpha)$ is analytic on $(\alpha(0), \alpha_{\sup})$, $(\alpha(1), \alpha(0))$, $(\alpha(2), \alpha(1))$,...,$(\alpha(N), \alpha(N-1))$,.... The phase transitions for $f_\mu(\alpha)$ are at $\alpha(1)$, $\alpha(2)$,...,$\alpha(N)$,.... The multifractal spectrum increases on $(\alpha_{\inf}, \alpha(0))$,

$$f_\mu(\alpha) = T(N) + N\alpha$$

on $(\alpha(N), \alpha(N-1))$ for each $N \in \mathbb{N}$, and finally, it decreases on $(\alpha(0), \alpha_{\sup})$. This example shows that the existence of $\alpha_{\lim}$ is necessary for obtaining Theorems 3.1.5 and 3.1.6.

CHAPTER 4

Large Deviations for an Expanding, Transient Map

4.1 Introduction

We consider an expanding, countably-branched Markov map on the interval. This dynamical systems has transient behaviour because orbits of Lebesgue typical points accumulate at 0. We chose to find a large deviation principle for this dynamical system because of this transience. Because of this transient behaviour, our rate function is the product of a weight function and a conditional variational principle. Typically, rate functions in large deviation principles do not have weight functions. We decided to work on the map's Markov shift and to form our large deviation principle on this space.

4.1.1 Setting of Our Problem

We introduce the dynamical system for our large deviation principle (see Theorem 4.1.6). Fix $\lambda \in \left(\frac{1}{2}, 1\right)$. Consider the map $T_\lambda : (0,1] \to (0,1]$:

$$T_\lambda(x) := \begin{cases} \frac{x-\lambda}{1-\lambda} & \text{if } x \in (\lambda, 1] \\ \frac{x-\lambda^n}{\lambda(1-\lambda)} & \text{if } x \in (\lambda^n, \lambda^{n-1}] \text{ for each } n \geq 2. \end{cases} \tag{4.1.1}$$

This map has a Markov partition $\{R_1, R_2, \ldots\}$ such that $R_n := (\lambda^n, \lambda^{n-1}]$ for each $n \in \mathbb{N}$. Stratmann and Vogt (see [SV97]) find dimension theoretic properties associated to this map. Bruin and Todd [BT12] analyse the thermodynamic properties of this map. The map T_λ is an expanding Markov map that is modelled by a countable state Markov shift Σ_A. The following graph of T_λ is taken from the first page of Bruin and Todd's paper [BT12].

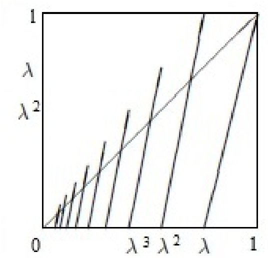


Figure 4.1: The Map T_λ

Define the transition matrix $A = (a_{i,j})$ as follows:

$$a_{i,j} = \begin{cases} 1 & \text{if } j \geq i-1 \text{ for } i \geq 2 \text{ or } j \geq i \text{ for } i = 1 \\ 0 & otherwise. \end{cases} \tag{4.1.2}$$

Let

$$\Sigma_A := \left\{ x \in \mathbb{N}^{\mathbb{N}} : a_{x_i, x_{i+1}} = 1 \text{ for every } i \geq 1 \right\}. \tag{4.1.3}$$

Take the typical notation for a cylinder set

$$[x_1, ..., x_n] = \{y \in \Sigma_A : y_i = x_i \text{ for each } 1 \leq i \leq n\}$$

and denote

$$[x_i, ..., x_n]$$

as the $(n-i+1)-$cylinder given by the last $(n-i+1)$ symbols of the cylinder $[x_1, ..., x_i, ..., x_n] \subset \Sigma_A$. Let $\sigma : \Sigma_A \to \Sigma_A$ be the left shift and $\pi : \Sigma_A \to (0,1]$ be the coding map such that $\pi^{-1}(R_n) = [n]$. In particular, there is a conjugacy (up to a countable number of points) between T_λ and σ :

$$T_\lambda \circ \pi = \pi \circ \sigma. \tag{4.1.4}$$

Consider the potential $\phi_\lambda := -\log |T_\lambda^{'} \circ \pi|$ and a locally Hölder potential $f : \Sigma_A \to \mathbb{R}$. Denote l for Lebesgue measure. We will take $m = l \circ \pi$ as the reference measure for the large deviation principle for $\frac{S_n f}{n}$.

Definition 4.1.1. *A measure m on Σ_A is* conformal *for σ if $m(\sigma(A)) = \int_A e^{\phi_\lambda} \, \mathrm{d}m$ whenever A is measurable and σ is injective on A.*

By Theorem 4.2.1, $m = l \circ \pi$ is a conformal measure. This property is key for our measure m. Hence, by Bruin and Todd's calculations (from Claim 1 and the proof of Theorem A in [BT12]),

$$m([x_1, ..., x_n]) = |T_\lambda^{'} \circ \pi([x_1])| m([x_2, ..., x_n])$$

$$= \cdots = |T_\lambda^{'} \circ \pi([x_1])| \cdots |T_\lambda^{'} \circ \pi([x_{n-1}])| m([x_n])$$

$$= |T_\lambda^{'} \circ \pi([x_1])| \cdots |T_\lambda^{'} \circ \pi([x_{n-1}])| \lambda^{x_n - 1}(1-\lambda). \tag{4.1.5}$$

for each $[x_1,...,x_n] \subset \Sigma_A$.

Equation (4.1.5) will help us bound $m([x_1,...,x_n])$ above (see Proposition 4.2.2). This estimate will be key for our results on pressure (see Section 4.6). Before stating the full setting for our large deviation principle, we analyse a property of our map T_λ. First, we define the escaping set as given on Page 173 of Bruin and Todd [BT12].

Definition 4.1.2. *The* escaping set

$$\Omega_\lambda := \{x \in \Sigma_A : \lim_{n\to\infty} T_\lambda^n \circ \pi(x) = 0\}.$$

We find that $\Sigma_A = \Omega_\lambda \cup \Omega_\lambda^{\complement}$. We will state that a sequence has transient behaviour or is transient if it is not recurrent (see Definition 2.2.5). In this case, if a sequence $x \in \Sigma_A$ is transient, there does not exist a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that

$$\sigma^{n_i}(x) \in [k]$$

for any $1-$cylinder $[k]$. This lemma will help us calculate $m(\Omega_\lambda)$. Bruin and Todd [BT12] prove this result about the orbits of Lebesgue typical points.

Lemma 4.1.3. *Fix $\lambda \in \left(\frac{1}{2}, 1\right)$. Then,*

$$\lim_{n\to\infty} T_\lambda^n(y) = 0$$

for Lebesgue-a.e. $y \in (0,1)$.

Proof. See the proof of Theorem 1 in Bruin and Todd [BT12]. ∎

We can now calculate $m(\Omega_\lambda)$. This theorem will be key for proving our large deviation principle (see Theorem 4.1.6). Bruin and Todd [BT12] show this result in the proof of Theorem 1 in their paper.

Theorem 4.1.4. *We find that $m(\Omega_\lambda) = 1$ and $m(\Omega_\lambda^{\complement}) = 0$. Take any $m-$typical $x \in \Sigma_A$. For each $k \in \mathbb{N}$, there does not exist a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that*

$$\sigma^{n_i}(x) \in [k].$$

Therefore, $m-$typical sequences are transient.

Proof. Fix $\lambda \in \left(\frac{1}{2}, 1\right)$. Lemma 4.1.3 states that

$$\lim_{n\to\infty} T_\lambda^n(y) = 0$$

for each Lebesgue typical $y \in (0,1)$. The set $\pi(\Omega_\lambda)$ contains this set of y. Hence,

$$l \circ \pi(\Omega_\lambda) = m(\Omega_\lambda) = 1 \text{ and } m(\Omega_\lambda^{\complement}) = 0. \tag{4.1.6}$$

By definition of the escaping set (see Definition 4.1.2)

$$\Omega_\lambda := \{x \in \Sigma_A : \lim_{n\to\infty} T_\lambda^n \circ \pi(x) = 0\},$$

we find the following. Consider an m−typical sequence $x \in \Sigma_A$, so $x \in \Omega_\lambda$ by Equation (4.1.6). Then, for each $k \in \mathbb{N}$, there does not exist a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that

$$\sigma^{n_i}(x) \in [k]$$

because $\pi(R_i) = \pi(\lambda^i, \lambda^{i-1}] = [i]$ and the definition of Ω_λ. Therefore, m−typical sequences are transient. ■

Hence, we will use that m−typical sequences are transient (see Theorem 4.1.4) in our large deviation argument.

4.1.2 Potentials for Our Large Deviation Problem and Methodology

Now, we will fully state necessary conditions for our potentials and discuss the methodology to form our large deviation principle (see Theorem 4.1.6). Let $\pi : \Sigma_A \to (0,1]$ be the coding map. In particular, there is a conjugacy (up to a countable number of points) between T_λ and σ :

$$T_\lambda \circ \pi = \pi \circ \sigma.$$

Let $\bar{N} := (N,N,N,...)$ for each $N \in \mathbb{N}$. Take a locally Hölder function (see Definition 2.3.17) $f : \Sigma_A \to \mathbb{R}$ such that

$$\lim_{N\to\infty} f(\bar{N}) \in (-\infty,\infty) \tag{4.1.7}$$

and $\phi_\lambda := -\log|T_\lambda \circ \pi|$. Denote $L := \lim_{N\to\infty} f(\bar{N})$. We will later (see Proposition 4.1.5) prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i(x)) = L$$

for m−a.e. $x \in \Omega_\lambda$. There exist values

$$\alpha_{\sup} := \sup_{\nu\in M_\sigma(\Sigma_A)} \left\{\int f \, d\nu\right\} \text{ and } \alpha_{\inf} := \inf_{\nu\in M_\sigma(\Sigma_A)} \left\{\int f \, d\nu\right\}. \tag{4.1.8}$$

We fix an $\alpha \in (L, \alpha_{\sup})$ to form our large deviation principle for $\frac{S_n f}{n}$ (see Theorem 4.1.6). Then, consider the set

$$X_\alpha^n := \left\{x \in \Sigma_A : \frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i(x)) \geq \alpha\right\} \tag{4.1.9}$$

for each $n \in \mathbb{N}$. We will aim to form an expression for the function R (see Theorem 4.1.6) such that

$$\lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) = R(\alpha)$$

for each $\alpha \in (L, \alpha_{\sup})$.

To construct this rate function, we need to construct subsets of X_α^n. Theorem 4.1.4 and Proposition 4.1.5, about the transient behaviour of our dynamical system and the Birkhoff averages of typical points, will help us construct these subsets.

Proposition 4.1.5. *Assume that $f : \Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $L = \lim_{N\to\infty} f(\bar{N}) \in (-\infty, \infty)$ and consider the potential $\phi_\lambda := -\log|T_\lambda' \circ \pi|$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i(x)) = L \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_\lambda(\sigma^i(x)) = \log[\lambda(1-\lambda)] \tag{4.1.10}$$

for each $m-$typical $x \in \Sigma_A$.

Proof. Because Theorem 4.1.4 states that $m(\Omega_\lambda) = 1$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i(x)) = \lim_{N\to\infty} f(\bar{N}) = L$$

for $m-$a.e. $x \in \Sigma_A$. The result for ϕ_λ follows from the same reasoning. ■

We will now state our large deviation principle. Our rate function uses a conditional variational principle, so we define a function by using this type of function. Define the function I as

$$I(\gamma) := \sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, d\eta + h(\eta) : \int f \, d\eta \ge \gamma \right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$. We also need to define our weight function, $p(\alpha)$ (see Equations (4.1.12), (4.1.13), and (4.1.14)). Consider the value

$$p_{\inf} := \frac{\alpha - L}{\alpha_{\sup} - L} \tag{4.1.11}$$

Define the function β as

$$\beta(p, \alpha) := \frac{\alpha - (1-p)L}{p} \tag{4.1.12}$$

for each $p \in (p_{\inf}, 1]$. Consider the values $p(\alpha) \in (p_{\inf}, 1]$ and $\beta(p(\alpha), \alpha) := \beta(\alpha) \in (\alpha, \alpha_{\sup})$ such that

$$\beta(\alpha) = \frac{\alpha - (1-p(\alpha))L}{p(\alpha)} \tag{4.1.13}$$

and

$$\max_{p_{\inf} < p \le 1} pI(\beta(p, \alpha)) = p(\alpha)I(\beta(\alpha)). \tag{4.1.14}$$

Theorem 4.1.6. *Fix* $\lambda \in \left(\frac{1}{2},1\right)$. *Recall the map* T_λ *given by Equation* (4.1.1) *and the shift space* (Σ_A,σ). *Let* $\phi_\lambda := -\log|T_\lambda^{'} \circ \pi|$. *Take* $\bar{N} := (N,N,N,...) \in \Sigma_A$ *for each* $N \in \mathbb{N}$. *Assume that* $f : \Sigma_A \to \mathbb{R}$ *is a locally Hölder potential such that* $\lim_{N\to\infty} f(\bar{N}) \in (-\infty,\infty)$. *Recall that*

$$L := \lim_{N\to\infty} f(\bar{N}) \text{ and } \alpha_{\sup} := \sup_{\nu\in M_\sigma(\Sigma_A)} \left\{ \int f \,\mathrm{d}\nu \right\}.$$

Fix $\alpha \in \left(L,\alpha_{\sup}\right)$. *Then, there exists a function* R, *defined by a* $p(\alpha) \in (p_{\inf},1]$ *and* $\beta(\alpha) \in (\alpha,\alpha_{\sup})$ *(see Equations* (4.1.12), (4.1.13), *and* (4.1.14)*), such that*

$$\lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) = R(\alpha) = p(\alpha)\left(\sup_{\eta\in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \,\mathrm{d}\eta + h(\eta) : \int f \,\mathrm{d}\eta \geq \beta(\alpha) \right\} \right) < 0.$$

Furthermore,

$$\lim_{\alpha\to L^+} R(\alpha) = 0.$$

To prove our large deviation principle, Theorem 4.1.6, we form the following bounds. We will first show (see Theorem 4.7.6) the upper bound: there exist $p(\alpha) \in (p_{\inf},1]$ and $\beta(\alpha) \in (\alpha,\alpha_{\sup})$ (given by Equations (4.1.13) and (4.1.14)) such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \leq p(\alpha) I(\beta(\alpha)).$$

To do this, we construct a subset of X_α^n. The subset will be formed from a recurrent set and a set whose symbols are quite large. From this subset, we construct periodic points and use Gurevich pressure (see Definition 2.3.22).

Then, we will show (see Theorem 4.8.6) the lower bound: there exist $p(\alpha) \in (p_{\inf},1]$ and $\beta(\alpha) \in (\alpha,\alpha_{\sup})$ (given by Equations (4.1.13) and (4.1.14)) such that

$$\liminf_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \geq p(\alpha) I(\beta(\alpha)).$$

To do this, we will construct a subset of X_α^n by using Egoroff's Theorem and the Birkhoff averages of $m-$typical sequences (see Proposition 4.1.5). Because $m-$typical sequences are transient (see Theorem 4.1.4), we will use this behaviour to motivate the formation of subsets of X_α^n in the proofs of the upper and lower bounds.

Hence, we now address the possibility that $L = -\infty$. The proof for the following large deviation principle is similar to the proof for Theorem 4.1.6. We would use a small value $K < 0$ rather than L to form subsets for X_α^n. There exists a function R such that

$$\lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) = R(\alpha) = \sup_{\eta\in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \,\mathrm{d}\eta + h(\eta) : \int f \,\mathrm{d}\eta \geq \alpha \right\} < 0 \tag{4.1.15}$$

for each $\alpha \in (L,\alpha_{\sup})$. Inequality (4.1.15) is the standard large deviation principle given by a conditional variational principle. For simplicity, we decided to form our large deviation principle on the shift. However, a large deviation principle, similar to Theorem 4.1.6, can be found on the interval (see Section 4.10 for a discussion).

4.1.3 Outline of Chapter

We give a guide to this chapter's contents. First, Section 4.2 proves necessary results about m and ϕ_λ. Section 4.3 constructs subsets of X_α^n for the upper bound of our large deviation principle. Then, the following sections form the groundwork for the upper bound. We form the upper bound of our large deviation principle, i.e., we will bound

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n)$$

above in Section 4.7. In Section 4.8, we form a lower bound of our large deviation principle, i.e., we will bound

$$\liminf_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n)$$

below. A discussion of the methods used to form these bounds was given after the statement of our large deviation principle (see the two paragraphs after Theorem 4.1.6). Finally, Section 4.9 combines the lower and upper bound to prove our large deviation principle, Theorem 4.1.6.

We will now state results, on ϕ_λ and the measure of cylinder sets, that will help us form our large deviation principle, Theorem 4.1.6.

4.2 Properties of ϕ_λ

In this section, we state results about ϕ_λ and estimates for the measure of $n-$cylinders. Fix $\lambda \in (0,1)$ in this section. Consider the map T_λ and its coding map $\pi : \Sigma_A \to (0,1]$ and shift $\sigma : \Sigma_A \to \Sigma_A$. We will state results on the thermodynamic formalism of our potential $\phi_\lambda = -\log|T_\lambda^{'} \circ \pi|$.

Bruin and Todd [BT12] state the following result on the thermodynamic formalism of ϕ_λ.

Theorem 4.2.1. *Recall the potential $\phi_\lambda = -\log|T_\lambda^{'} \circ \pi|$. Thus,*

$$\mathscr{P}(\phi_\lambda) = \begin{cases} 0 & \text{if } \lambda \le \frac{1}{2}; \\ \log[4\lambda(1-\lambda)] & \text{if } \lambda \ge \frac{1}{2}. \end{cases}$$

At least one conformal measure, including $m = l \circ \pi$, for σ exists. Furthermore, there does not exist an equilibrium state for ϕ_λ when $\lambda \in (\frac{1}{2}, 1)$.

Proof. See Theorem A from Bruin and Todd [BT12]. ∎

Recall that we aim to find an expression (see Theorem 4.1.6) for a rate function R such that

$$R(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n)$$

for each $\alpha \in (L, \alpha_{\text{sup}})$ (see Equation (4.1.9) for the definition of the set X_α^n). We will use results about the pressure of ϕ_λ (see Theorem 4.2.1) and the behaviour of $m-$typical points (see Theorem 4.1.4 and Proposition 4.1.5) to form this expression.

Before forming our large deviation principle (see Theorem 4.1.6), we will prove necessary estimates on the measure of $n-$cylinders.

4.2.1 Estimating the Measure of Cylinders

Because m is conformal, we obtain the following bounds for the $m-$measure of $n-$cylinders.

Proposition 4.2.2. *For each cylinder* $[x_1,...,x_{n-1},x_n] \subset \Sigma_A$ *and every* $x \in [x_1,...,x_{n-1},x_n]$, *we find that*

$$m([x_1,...,x_{n-1},x_n]) \le \exp\left(S_n\phi_\lambda(x)\right).$$

Proof. Consider an arbitrary $n-$cylinder $[x_1,...,x_{n-1},x_n] \subset \Sigma_A$ and choose any $x \in [x_1,...,x_{n-1},x_n]$.

We will now find expressions for $m([x_1,...,x_{n-1},x_n])$ and $\exp(S_n\phi_\lambda(x))$. First, we find an expression for $m([x_1,...,x_{n-1},x_n])$. Define

$$N([x_1,...,x_n]) := n - \sum_{i=0}^{n-1} \mathbb{1}_{[1]}(\sigma^i(x)). \tag{4.2.1}$$

In other terms, this function gives the number of symbols x_i, such that $1 \le i \le n$, of an $n-$cylinder that are greater than 1. Then,

$$m([x_1,...,x_{n-1},x_n]) = \lambda^{N([x_1,...,x_{n-1}])}(1-\lambda)^{n-1}m([x_n])$$

$$= \lambda^{N([x_1,...,x_{n-1}])}(1-\lambda)^{n-1}\lambda^{x_n-1}(1-\lambda) = \lambda^{N([x_1,...,x_{n-1}])+x_n-1}(1-\lambda)^n$$

by Equation (4.1.5). Hence,

$$m([x_1,...,x_{n-1},x_n]) = \lambda^{N([x_1,...,x_{n-1}])+x_n-1}(1-\lambda)^n. \tag{4.2.2}$$

Now, we will find an expression for $\exp\left(S_n\phi_\lambda(x)\right)$. Because $\phi_\lambda := -\log|T_\lambda' \circ \pi|$,

$$\exp\left(\sum_{i=0}^{n-1}\phi_\lambda(\sigma^i(x))\right) = \exp\left(-\sum_{i=0}^{n-1}\log|T_\lambda'(T_\lambda^i \circ \pi(x))|\right) = \left(\prod_{i=0}^{n-1}|T_\lambda'(T_\lambda^i \circ \pi(x))|\right)^{-1}.$$

The conjugacy (up to a countable number of points) between T_λ and σ gives us that

$$T_\lambda^i \circ \pi([x_1,...,x_n]) = \pi([x_{i+1},...,x_n])$$

for each $1 \le i \le n-1$. Since $T_\lambda^i(\pi(x)) \in \pi([x_{i+1}])$ and $T_\lambda' \circ \pi(x)$ equals $\frac{1}{1-\lambda}$ or $\frac{1}{\lambda(1-\lambda)}$ depending on the first symbol of $x = (x_1,...,x_n,...) \in \Sigma_A$,

$$\lambda^{N([x_1,...,x_n])}(1-\lambda)^n = \left(\prod_{i=0}^{n-1}|T_\lambda'(T_\lambda^i \circ \pi(x))|\right)^{-1} = \exp\left(\sum_{i=0}^{n-1}\phi_\lambda(\sigma^i(x))\right). \tag{4.2.3}$$

Let us now prove that

$$m([x_1,...,x_{n-1},x_n]) \le \exp\left(S_n\phi_\lambda(x)\right).$$

We have the inequality

$$\lambda^{N([x_1,...,x_{n-1}])+x_n-1}(1-\lambda)^n \le \lambda^{N([x_1,...,x_n])}(1-\lambda)^n$$

by construction of the function N given by Equation (4.2.1). Hence, by combining Equations (4.2.2) and (4.2.3),

$$m([x_1,...,x_{n-1},x_n]) \leq \exp\left(S_n\phi_\lambda(x)\right)$$

for any $x \in [x_1,...,x_{n-1},x_n]$. Our choice of cylinder $[x_1,...,x_{n-1},x_n] \subset \Sigma_A$ was arbitrary, so the result follows. ∎

We will also need the following lemma to form a large deviation principle for $\frac{S_n f}{n}$.

Lemma 4.2.3. *For each cylinder $[x_1,...,x_k,...,x_n] \subset \Sigma_A$, we find that*

$$m([x_1,...,x_k,...,x_n]) \geq m([x_1,...,x_k])m([x_{k+1},...,x_n]).$$

Proof. First, we find that

$$m([x_1,...,x_k])m([x_{k+1},...,x_n]) = \lambda^{N([x_1,...,x_{k-1}])+x_k-1}(1-\lambda)^k\lambda^{N([x_{k+1},...,x_{n-1}])+x_n-1}(1-\lambda)^{n-k}$$

$$= \lambda^{N([x_1,...,x_{k-1}])+N([x_{k+1},...,x_{n-1}])+x_k-1+x_n-1}(1-\lambda)^n \tag{4.2.4}$$

Let $1 \leq i \leq n-1$. We have that $x_i - 1 \geq 1$ if $x_i \geq 2$. Hence,

$$\lambda^{N([x_1,...,x_{n-1}])+x_n-1}(1-\lambda)^n \geq \lambda^{N([x_1,...,x_{k-1}]+N([x_{k+1},...,x_{n-1}])+x_k-1+x_n-1}(1-\lambda)^n.$$

Combined with Equations (4.2.2) and (4.2.4), we get our result

$$m([x_1,...,x_k,...,x_n]) \geq m([x_1,...,x_k])m([x_{k+1},...,x_n]).$$

∎

We will now summarise the setting of our large deviations problem and form the upper bound (see Theorem 4.7.6) for our large deviation principle.

4.3 Revisiting the Setting and Outlining of our Upper Bound's Proof

First, we revisit our large deviations problem's setting. Consider $\bar{N} := (N,N,N,...)$ for each $N \in \mathbb{N}$. Assume that $f : \Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty,\infty)$. Recall that

$$L := \lim_{N\to\infty} f(\bar{N}) \text{ and } \alpha_{\sup} := \sup_{\nu\in M_\sigma(\Sigma_A)} \left\{\int f \,\mathrm{d}\nu\right\}.$$

Fix $\alpha \in (L,\alpha_{\sup})$. For each $n \in \mathbb{N}$, define the set

$$X_\alpha^n := \left\{x \in \Sigma_A : \frac{1}{n}\sum_{i=0}^{n-1} f(\sigma^i(x)) \geq \alpha\right\}.$$

We use this set to form our large deviation principle, Theorem 4.1.6.

We will first bound

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n)$$

above (see Theorem 4.7.6). To find this bound, we will later find an upper bound for $m(X_\alpha^n)$ (see Theorem 4.7.2). To do this, we follow this sequence of steps.

1. First, we construct a set $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor} \subset X_\alpha^n$ (see Equations (4.4.4) and (4.4.5)) in Section 4.4. This set is defined by the hitting times of its elements on $\cup_{i=1}^M [i]$. Our choice of M (see Inequality (4.4.2)) is key because we will use it to prove that the union of these subsets is X_α^n (see Proposition 4.4.1). Hence, to bound $m(X_\alpha^n)$ above, we will find an upper bound for $m(X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor})$. To do this, we will construct subsets of $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$. For simplicity, we will consider the set $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ for an arbitrary $k \in \{1,..,r-1,r\}$.

2. Recall that $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ is defined by the hitting times of its elements on the set $\cup_{i=1}^M [i]$. We will use these hitting times to express $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ as a union of subsets. Consider an arbitrary $j(k) \in (k-1,k]$. Then, we construct a set $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \subset X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$. For each $x = (x_1,...,x_{j(k)},...,x_n) \in \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$, $j(k) = \max\{1 \le i \le n : x_i \le M\}$. We will later (see Equation (4.4.14)) find that

$$X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor} = \bigcup_{j\in(k-1,k]} \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}.$$

   Hence, we will bound $m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right)$ above to find an upper bound for $m(X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor})$. To find this upper bound, we construct a set of cylinders that contain periodic points. In turn, we will use pressure to find this upper bound.

3. Using the set $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$, we will construct a set of cylinders $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(1)$ that contain periodic points. This will be done in Section 4.5.

4. Because we will use these periodic points to calculate Gurevich pressure, we prove results on Gurevich pressure in Section 4.6. In one of these results (see Proposition 4.6.3), we bound Gurevich pressure above by a conditional variational principle.

5. In Section 4.7, we combine the results on $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$, the constructed set of cylinders $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(1)$ that contain periodic points, and Gurevich pressure from Sections 4.4-4.6. We find an upper bound for $m(X_\alpha^n)$ (see Theorem 4.7.2). This eventually leads to the upper bound for

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n)$$

   in Theorem 4.7.6.

Now, we will construct subsets of X_α^n.

4.4 Constructing Subsets of X_α^n

To bound $m(X_\alpha^n)$ above for each $n \in \mathbb{N}$, we will construct a class of subsets for X_α^n. Because $m-$typical sequences are transient (see Theorem 4.1.4), we will use these sequences and their Birkhoff averages to motivate the construction of these subsets. First, we recall a few results and define the variables for these sets.

For each $N \in \mathbb{N}$, consider the sequence $\bar{N} = (N, N, N, \ldots)$. Denote $L := \lim_{N\to\infty} f(\bar{N})$. We found that

$$\lim_{n\to\infty} \frac{S_n f(x)}{n} = L \tag{4.4.1}$$

for $m-$typical $x \in \Sigma_A$ by Theorem 4.1.4. This will motivate our choice of $M \in \mathbb{N}$ (see Inequality (4.4.2)).

Let $\varepsilon \in (0, \alpha - L)$. For each $J \in \mathbb{N}$, consider the $1-$cylinder $[J]$. Take any sequence $\widehat{J} := (J, x_2, x_3, \ldots) \in [J]$ for each $J \in \mathbb{N}$. There exists an $M := M(\varepsilon)$ such that for all $J \geq M$,

$$f(\widehat{J}) \leq L + \varepsilon. \tag{4.4.2}$$

We use this value of M to form a class of subsets of X_α^n (see Equations (4.4.4) and (4.4.5)). Throughout Sections 4.4-4.7, we will assume that our choice of M satisfies Inequality (4.4.2). Because M satisfies Inequality (4.4.2), we will prove that the union of these subsets forms the set X_α^n (see Proposition 4.4.1).

Now, we will construct the class of subsets of X_α^n. We take a subset of $[0,1]$ to construct our desired subsets. Fix an arbitrarily large $r \in \mathbb{N}$. Let

$$K_r := \left\{\frac{1}{r}, \frac{2}{r}, \ldots, \frac{r-2}{r}, \frac{r-1}{r}, 1\right\}. \tag{4.4.3}$$

Hence, for each $k \in \{1, \ldots, r-1, r\}$, define the set

$$X_M^{n,\lfloor n\frac{k}{r}\rfloor} := \tag{4.4.4}$$

$$\left\{x \in \Sigma_A : \exists j \in \left[\left\lfloor n\frac{(k-1)}{r}\right\rfloor + 1, \left\lfloor n\frac{k}{r}\right\rfloor\right] \text{ such that } x_j \leq M \text{ and } x_i \geq M+1 \ \forall i \in \left[\left\lfloor n\frac{k}{r}\right\rfloor + 1, n\right]\right\}.$$

We use $X_M^{n,\lfloor n\frac{k}{r}\rfloor}$ to construct subsets of X_α^n. We now define this class of subsets:

$$X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor} := X_M^{n,\lfloor n\frac{k}{r}\rfloor} \cap X_\alpha^n. \tag{4.4.5}$$

We will prove that union of these subsets forms X_α^n (see Proposition 4.4.1). By Equation (4.4.4),

$$X_M^{n,n} := \left\{x \in \Sigma_A : \exists j \in \left[\left\lfloor n\frac{(r-1)}{r}\right\rfloor + 1, n\right] \text{ such that } x_j \leq M\right\}. \tag{4.4.6}$$

Take

$$X_{\alpha,M}^{n,n} := X_M^{n,n} \cap X_\alpha^n. \tag{4.4.7}$$

For each $y = (y_1, ..., y_n) \in \cup_{k=1}^{r} X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$, there exists a value

$$y_{\inf} := \inf\{i \in [1,n] : y_i \leq M\} \tag{4.4.8}$$

by definition of $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ (see Equation (4.4.4) and (4.4.6)).

Now, we will prove that $X_\alpha^n = \cup_{k=1}^{r} X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$. This is why the value M from Inequality (4.4.2) is key.

Proposition 4.4.1. *We find that*

$$X_\alpha^n = \cup_{k=1}^{r} X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}.$$

Proof. Take a sequence $x = (x_1, ..., x_n) \notin \cup_{k=1}^{r} X_{M}^{n,\lfloor n\frac{k}{r}\rfloor}$. Then,

$$x_i \geq M+1 \tag{4.4.9}$$

for all $i \in \{1, ..., n\}$. Take the $1-$cylinder $[J]$ for each $J \in \mathbb{N}$. Consider the sequence $\widehat{J} := (J, x_2, x_3, ...) \in [J]$. Let $\varepsilon \in (0, \alpha - L)$. Recall the condition for our choice of M (see Inequality (4.4.2)): there exists a $M := M(\varepsilon)$ such that for all $J \geq M$,

$$f(\widehat{J}) \leq L + \varepsilon.$$

Hence,

$$\frac{S_n f(x)}{n} = \frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i(x)) \leq L + \varepsilon < \alpha \tag{4.4.10}$$

by Inequalities (4.4.9) and (4.4.2). Then,

$$x \notin X_\alpha^n \tag{4.4.11}$$

by Inequality (4.4.10). Therefore,

$$X_\alpha^n = \cup_{k=1}^{r} X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$$

by Inclusion (4.4.11). ■

Recall that we aim to find an upper bound for

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n).$$

We proved that

$$X_\alpha^n = \cup_{k=1}^{r} X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}.$$

Hence, for each $k \in \{1, ..., r\}$, we will bound

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor})$$

above. For simplicity, we will consider $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ for an arbitrary $k \in \{1, ..., r\}$.

To find an upper bound for

$$\limsup_{n\to\infty}\frac{1}{n}\log m(X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}),$$

we will bound $m\left(X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}\right)$ above. To find the upper bound for the measure of $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$, we will consider a class of sets $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}\subset X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$. These subsets are defined by hitting times on $\cup_{i=1}^M[i]$, so they contain recurrent sequences. For this reason, we will later construct a set of periodic points from $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}$ in Section 4.5.

4.4.1 A Subset $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}$ of X_α^n

Now, we will construct a class of subsets $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}\subset X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}\subset X_\alpha^n$ by using hitting times. We now give motivation for this class of subsets. Recall that $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}=X_M^{n,\lfloor n\frac{k}{r}\rfloor}\cap X_\alpha^n$. Take an

$$x=(x_1,\dots,x_{\lfloor n\frac{k}{r}\rfloor},\dots,x_n)\in X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}.$$

There exists a $j\in(k-1,k]$ such that $x_{\lfloor n\frac{j}{r}\rfloor}\le M$ and $x_{\lfloor n\frac{j}{r}\rfloor+1}>M$. Hence, we will express $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ as a union of the following sets.

For each $j\in(k-1,k]$, take the set

$$\widehat{X}_M^{n,\lfloor n\frac{j}{r}\rfloor}:=\left\{x\in\Sigma_A : x_{\lfloor n\frac{j}{r}\rfloor}\le M \text{ and } x_i\ge M+1\ \forall i\in\left[\left\lfloor n\frac{j}{r}\right\rfloor+1,n\right]\right\}. \tag{4.4.12}$$

As seen above, the set $\widehat{X}_M^{n,\lfloor n\frac{j}{r}\rfloor}$ is defined by hitting times on the set $\cup_{i=1}^M[i]$. Consider

$$\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}:=\widehat{X}_M^{n,\lfloor n\frac{j}{r}\rfloor}\cap X_\alpha^n. \tag{4.4.13}$$

Naturally, we form the union:

$$X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}=\bigcup_{j\in(k-1,k]}\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor} \tag{4.4.14}$$

We will later bound $m(X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor})$ above by using one of the $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}$ (see Equation (4.7.18) and Inequality (4.7.20)). For now, we take an arbitrary $j(k)\in(k-1,k]$ and in turn, we consider the set $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$. For each $x=(x_1,\dots,x_{j(k)},\dots,x_n)\in\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$, $j(k)=\max\{1\le i\le n : x_i\le M\}$. In Section 4.5, we will be able to construct a set of cylinders that contain periodic points because each $x=(x_1,\dots,x_{\lfloor n\frac{j(k)}{r}\rfloor},\dots)\in\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$ satisfies $x_{\lfloor n\frac{j(k)}{r}\rfloor}\le M$. First, we proceed by covering $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$ because a cover will help us build periodic points, use Gurevich pressure, and form a weighted conditional variational principle for the upper bound for our large deviation principle, Theorem 4.7.6.

4.4.2 Covering $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$

We will prove that the following set covers $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$ for a $\beta \in (\alpha, \alpha_{\sup})$.

For a $\beta \in (\alpha, \alpha_{\sup})$, define the set

$$\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor} := \widehat{X}_{M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \cap X_{\beta}^{\lfloor n\frac{j(k)}{r}\rfloor}.$$

The following proposition determines our choice of β and proves that $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor} \supset \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$. The cover is formed by $\lfloor n\frac{j(k)}{r}\rfloor-$cylinders that contain recurrent sequences. Hence, this cover will be key to our construction of periodic points and use of pressure.

Proposition 4.4.2. *Fix $\alpha \in (L, \alpha_{\sup})$. Let $0 < \varepsilon < \alpha - L$. There exists a value $\beta := \beta(\alpha, \varepsilon, j(k), r) \in (\alpha, \alpha_{\sup})$ such that*

1. $$\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \subset \widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor} \text{ and}$$

2. $$\frac{j(k)}{r}\beta(\alpha,\varepsilon,j(k),r) + \left(1 - \frac{j(k)}{r}\right)(L+\varepsilon) = \alpha$$

   or alternatively,

   $$\beta(\alpha,\varepsilon,j(k),r) = \frac{\alpha - \left(1 - \frac{j(k)}{r}\right)(L+\varepsilon)}{\frac{j(k)}{r}}.$$

Proof. Let $0 < \varepsilon < \alpha - L$. Consider an arbitrary $x = (x_1, ..., x_n) \in \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$. Then, for each $i \in \left\{\lfloor n\frac{j(k)}{r}\rfloor + 1, ..., n-1, n\right\}$,

$$x_i \geq M + 1. \tag{4.4.15}$$

Hence, by Inequality (4.4.15) and our condition for M (see Inequality (4.4.2)),

$$\frac{1}{n - \lfloor n\frac{j(k)}{r}\rfloor} \sum_{i=\lfloor n\frac{j(k)}{r}\rfloor}^{n-1} f(\sigma^i(x)) \leq L + \varepsilon \tag{4.4.16}$$

for all $x \in \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$.

Now, we will use Inequality (4.4.16) to characterise our choice of β. Because of Inequality (4.4.16), $x \in \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \subset X_\alpha^n$, and $\alpha \in (L, \alpha_{\sup})$, there exists a value $\beta := \beta(\alpha, \varepsilon, j(k), r) \in (\alpha, \alpha_{\sup})$ that satisfies

$$\frac{1}{\lfloor n\frac{j(k)}{r}\rfloor} \sum_{i=0}^{\lfloor n\frac{j(k)}{r}\rfloor - 1} f(\sigma^i(x)) \geq \beta(\alpha,\varepsilon,j(k),r) \text{ and} \tag{4.4.17}$$

$$\sum_{i=0}^{n-1} f(\sigma^i(x)) \geq \lfloor n\frac{j(k)}{r}\rfloor \beta(\alpha,\varepsilon,j(k),r) + (n - \lfloor n\frac{j(k)}{r}\rfloor)(L+\varepsilon) \tag{4.4.18}$$

for each $x \in \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$. Furthermore, $\beta := \beta(\alpha,\varepsilon,j(k),r) \in (\alpha,\alpha_{\sup})$ can be taken to satisfy

$$\frac{j(k)}{r}\beta(\alpha,\varepsilon,j(k),r) + \left(1 - \frac{j(k)}{r}\right)(L+\varepsilon) = \alpha \tag{4.4.19}$$

or alternatively,

$$\beta(\alpha,\varepsilon,j(k),r) = \frac{\alpha - \left(1-\frac{j(k)}{r}\right)(L+\varepsilon)}{\frac{j(k)}{r}}$$

by Equation (4.4.19).

This gives us that

$$\sum_{i=0}^{n-1} f(\sigma^i(x)) \geq \lfloor n\frac{j(k)}{r}\rfloor \beta(\alpha,\varepsilon,j(k),r) + (n - \lfloor n\frac{j(k)}{r}\rfloor)(L+\varepsilon) = n\alpha \tag{4.4.20}$$

for all $x \in \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$ by Inequalities (4.4.18) and (4.4.19).

Thus,

$$\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \subset \widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$$

by Inequalities (4.4.17) and (4.4.18) and Equation (4.4.19). ∎

As discussed earlier, we will bound $m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor})$ to bound $m(X_\alpha^n)$ above (see Theorem 4.7.2). We will use that $m(\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}) \geq m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor})$ (which immediately follows from Proposition 4.4.2). This will eventually help us form an upper bound (see Theorem 4.7.6)

$$\overline{R}(\alpha) \geq \limsup_{n\to\infty} \frac{1}{n}\log m(X_\alpha^n).$$

Now, we will use our cover $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$ to build a set of periodic points.

4.5 Forming An Upper Bound via Periodic Points

We will now use $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$ to build a set of cylinders that contain periodic points (see Subsection 4.5.1 for the procedure). By using these periodic points, we will find the Gurevich pressure of a potential and then, form a weighted conditional variational principle for our upper bound.

4.5.1 Plan for Constructing Periodic Points from $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$

We outline the steps used to construct periodic points from $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$.

1. We will consider the set $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}$ of $\lfloor n\frac{j(k)}{r}\rfloor$−cylinders in $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$.

2. For each element of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}$, there exists a $\left(\lfloor n\frac{j(k)}{r}\rfloor+M+1\right)-$cylinder $[1,x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]$ such that its symbols satisfy
$$x_{i+1}=x_i-1\geq 1$$
for each $i\in\{\lfloor n\frac{j(k)}{r}\rfloor,...,\lfloor n\frac{j(k)}{r}\rfloor+M-1\}$.

3. By using these cylinders, we form an upper bound for $m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor})$. This upper bound is a sum over $(\lfloor n\frac{j(k)}{r}\rfloor+M)-$periodic points that start with the symbol 1.

Take the set $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}$ of $\lfloor n\frac{j(k)}{r}\rfloor-$cylinders in $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$:

$$\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}:=\left\{[x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor}]\subset\Sigma_A:\exists x=(x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor},...)\in\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}\right\}. \tag{4.5.1}$$

4.5.2 Constructing a Subset of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}$

We will use the following set to construct a subset of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}$. Define the set

$$X_{M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}:=\left\{[x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor},...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M}]\subset\Sigma_A:x_{i+1}=x_i-1\geq 1\text{ for all }\lfloor n\frac{j(k)}{r}\rfloor\leq i\leq\lfloor n\frac{j(k)}{r}\rfloor+M\right\}.$$

Using this set, take

$$\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}:=\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}\cap X_{M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}. \tag{4.5.2}$$

We find a property of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$. For each element $[x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor},...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},x_{\lfloor n\frac{j(k)}{r}\rfloor+M}]\in\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$, its $(\lfloor n\frac{j(k)}{r}\rfloor+M)-$th symbol

$$x_{\lfloor n\frac{j(k)}{r}\rfloor+M}=1. \tag{4.5.3}$$

The property, given by Equation (4.5.3), is proven as follows. The $\lfloor n\frac{j(k)}{r}\rfloor-$th symbol $x_{\lfloor n\frac{j(k)}{r}\rfloor}$ of any $(\lfloor n\frac{j(k)}{r}\rfloor+M+1)-$cylinder $[x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor},...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M}]\in\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$ satisfies

$$x_{\lfloor n\frac{j(k)}{r}\rfloor}\leq M. \tag{4.5.4}$$

Then, it takes at most $M-1$ steps to go from the $\lfloor n\frac{j(k)}{r}\rfloor-$th symbol $x_{\lfloor n\frac{j(k)}{r}\rfloor}$ to 1 by construction of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$, our transition matrix (see Equation (4.1.2)), and the upper bound for $x_{\lfloor n\frac{j(k)}{r}\rfloor}$ (see Inequality (4.5.4)). Thus, the $(\lfloor n\frac{j(k)}{r}\rfloor+M)-$th symbol of $[x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M}]\in\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$ satisfies

$$x_{\lfloor n\frac{j(k)}{r}\rfloor+M}=1.$$

Our large deviation argument will require us to find an inequality between the $m-$measures of respective elements of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}$ and $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$.

Proposition 4.5.1. *Fix* $\alpha \in (L, \alpha_{\sup})$. *Consider*

$$C(M) := \left[\frac{1}{\lambda(1-\lambda)}\right]^M > 4$$

for each $M \in \mathbb{N}$. *For each cylinder* $[x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor}] \in \widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor, c}$, *there exists a* $(\lfloor n\frac{j(k)}{r}\rfloor + M)$*–cylinder* $[x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}, 1] \in \widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor, d} \cap [x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor}]$ *such that*

$$m([x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor}]) < C(M) m([x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}, 1]).$$

Furthermore, the sequence $\{C(M)\}_{M\in\mathbb{N}}$ *is unbounded.*

Proof. For each $\lfloor n\frac{j(k)}{r}\rfloor$–cylinder $[x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor}] \in \widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor, c}$, there exists a $(\lfloor n\frac{j(k)}{r}\rfloor + M)$–cylinder $[x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}, 1] \in \widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor, d} \cap [x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor}]$ because of our transition matrix (see Equation (4.1.2)). Hence, let us take a $\lfloor n\frac{j(k)}{r}\rfloor$–cylinder $[x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor}] \in \widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor, c}$ and one of its subsets $[x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}, 1] \in \widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor, d}$. Given any $x \in [x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}, 1]$,

$$\frac{m([x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor}])}{m([x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}, 1])} < \frac{\exp S_{\lfloor n\frac{j(k)}{r}\rfloor}\phi_\lambda(x)}{(1-\lambda)\exp S_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}\phi_\lambda(x)} \tag{4.5.5}$$

by Equation (4.2.3) and Proposition 4.2.2.

By construction of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor, d}$, Equation (4.2.3), and Equation (4.2.1),

$$\frac{\exp S_{\lfloor n\frac{j(k)}{r}\rfloor}\phi_\lambda(x)}{\exp S_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}\phi_\lambda(x)} = \frac{\lambda^{N([x_1,\ldots,x_{\lfloor n\frac{j(k)}{r}\rfloor}])}(1-\lambda)^{\lfloor n\frac{j(k)}{r}\rfloor}}{\lambda^{N([x_1,\ldots,x_{\lfloor n\frac{j(k)}{r}\rfloor},\ldots,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1}])}(1-\lambda)^{\lfloor n\frac{j(k)}{r}\rfloor + M-1}} \le \frac{(1-\lambda)^{\lfloor n\frac{j(k)}{r}\rfloor}}{\lambda^{M-1}(1-\lambda)^{\lfloor n\frac{j(k)}{r}\rfloor + M-1}}$$

$$= \left[\frac{1}{\lambda(1-\lambda)}\right]^{M-1}. \tag{4.5.6}$$

By Inequalities (4.5.5) and (4.5.6),

$$\begin{aligned} m([x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor}]) &< \frac{1}{1-\lambda}\left[\frac{1}{\lambda(1-\lambda)}\right]^{M-1} m([x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}, 1]) \\ &< \left[\frac{1}{\lambda(1-\lambda)}\right]^{M} m([x_1, \ldots, x_{\lfloor n\frac{j(k)}{r}\rfloor + M-1}, 1]). \end{aligned}$$

Then, take

$$C(M) := \left[\frac{1}{\lambda(1-\lambda)}\right]^M$$

for each $M \in \mathbb{N}$. We find that

$$\lim_{M\to\infty} C(M) = \infty$$

because

$$C(M) = \left[\frac{1}{\lambda(1-\lambda)}\right]^M > 4^M \ge 4$$

for each $M \in \mathbb{N}$. ∎

4.5.3 Constructing a Set Containing $\left(\lfloor n\frac{j(k)}{r}\rfloor + M\right)$−Periodic Points

Recall that the final symbol of each cylinder in $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$ is a 1. Hence, to construct a set containing periodic points, we concatenate the symbol 1 onto each cylinder in $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$. For each $(\lfloor n\frac{j(k)}{r}\rfloor + M)$−cylinder $[x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]\in\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$, there exists a cylinder $[1,x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]\in\Sigma_A$ because the transition matrix (see Equation (4.1.2)) has the entry $a_{1,i}=1$ for every $i\in\mathbb{N}$.

We define the following key set of cylinders to form $(\lfloor n\frac{j(k)}{r}\rfloor+M)$−periodic points from $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$.

$$Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1):=[1]\cap\sigma^{-1}(\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d})=[1]\cap\sigma^{-1}(\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}\cap X_{M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}).$$

These cylinders contain $(\lfloor n\frac{j(k)}{r}\rfloor+M)$−periodic points $x\in\Sigma_A$ that start with the symbol 1 because of the definition of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$ and the transition matrix A has the entry $a_{1,i}=1$ for every $i\in\mathbb{N}$. Hence, we define the following sets of periodic points. For each $l\in\mathbb{N}$, consider

$$Per_l(1):=\{x=(x_1,...,x_l,x_{l+1},...)\in\Sigma_A:\sigma^l(x)=x \text{ and } x_1=1\} \text{ and}$$

$$Per_{l,c}(1):=\{[y_1,...,y_l,y_{l+1}]\subset\Sigma_A:\text{there exists } y=(y_1,...,y_l,y_{l+1},...)\in Per_l(1)\}.$$

We prove an inclusion that relates $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$ to $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(1)$.

Proposition 4.5.2. *For each $(\lfloor n\frac{j(k)}{r}\rfloor+M)$−cylinder $[x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]\in\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$, there exists a $(\lfloor n\frac{j(k)}{r}\rfloor+M+1)$−cylinder $[1,x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]\in Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$ and conversely. Furthermore,*

$$Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)\subset Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(1).$$

Proof. The first statement follows from construction. The second statement is immediate from the definitions of $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$ and $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(1)$. ∎

Our large deviation argument relies on an inequality on the measures of respective elements of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$ and $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$.

Proposition 4.5.3. *Fix $\alpha\in(L,\alpha_{\sup})$. For each $(\lfloor n\frac{j(k)}{r}\rfloor+M)$−cylinder $[x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]\in\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$, there exists a $(\lfloor n\frac{j(k)}{r}\rfloor+M+1)$− cylinder $[1,x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]\in Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$ such that*

$$m([x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1])\le\frac{1}{1-\lambda}m([1,x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]).$$

Proof. Such a cylinder exists by Proposition 4.5.2. We find that

$$\frac{m([x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1])}{m([1,x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1])}\le\frac{m([x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1])}{m([1])m([x_1,...,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1])}=\frac{1}{1-\lambda}$$

by Lemma 4.2.3 and the construction of $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$ and $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$. ∎

Because we have now constructed a set $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$ of cylinders containing periodic points, we analyse the behaviour of pressure. This will later help us form the upper bound for our large deviation principle.

4.6 Pressure

Consider the value $\beta := \beta(\alpha,\varepsilon,j(k),r)$ from Proposition 4.4.2. We will relate the potential $\phi_\lambda + q(f-\beta)$ to the set of cylinders $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(1)$ that contain $(\lfloor n\frac{j(k)}{r}\rfloor + M)$ periodic points. Furthermore, these periodic points will help us use Gurevich pressure to bound $m(X_\alpha^n)$ above. To do this, we will analyse the limiting behaviour of the function $q \mapsto \mathscr{P}(\phi_\lambda + q(f-\beta))$, find the minimum of the function, and prove that this minimum is bounded above by a conditional variational principle. We will use the compact approximation of pressure to form this bound.

Take the values:

$$q_0 := \{q>0 : \mathscr{P}(\phi_\lambda + q(f-\beta)) = \mathscr{P}(\phi_\lambda)\} \text{ and } q_1 := \{q>0 : \mathscr{P}(\phi_\lambda + q(f-\beta)) \le 0\}. \tag{4.6.1}$$

The following proposition proves that $q_1 > 0$ exists and states conditions for q_0 to exist.

We need these results to use pressure in the proof of Theorem 4.7.6.

Proposition 4.6.1. *We find that*

$$\lim_{q\to\infty} \mathscr{P}(\phi_\lambda + q(f-\beta)) = \lim_{q\to-\infty} \mathscr{P}(\phi_\lambda + q(f-\beta)) = \infty.$$

Then, there exists a value $0 \le q^* < q_1 < \infty$ *such that*

$$\min_{q\ge 0} \mathscr{P}(\phi_\lambda + q(f-\beta)) = \mathscr{P}(\phi_\lambda + q^*(f-\beta)) \le \mathscr{P}(\phi_\lambda). \tag{4.6.2}$$

Proof. We first analyse the limiting behaviour of

$$q \mapsto \mathscr{P}(\phi_\lambda + q(f-\beta)).$$

Consider a measure $\nu \in M_\sigma(\Sigma_A)$ such that

$$\int f \,\mathrm{d}\nu > \beta. \tag{4.6.3}$$

Because of the variational principle and ϕ_λ is bounded,

$$\int \phi_\lambda \mathrm{d}\nu + q\left[\int f\,\mathrm{d}\nu - \beta\right] + h(\nu) \le \mathscr{P}(\phi_\lambda + q(f-\beta)). \tag{4.6.4}$$

Then,

$$\lim_{q\to\infty}\int \phi_\lambda \mathrm{d}\nu + q\left[\int f\,\mathrm{d}\nu - \beta\right] + h(\nu) = \lim_{q\to\infty} \mathscr{P}(\phi_\lambda + q(f-\beta)) = \infty. \tag{4.6.5}$$

by Inequalities (4.6.4) and (4.6.3).

Similarly, take a measure $\nu \in M_\sigma(\Sigma_A)$ such that

$$\int f \mathrm{d}\nu < \beta. \tag{4.6.6}$$

Then,

$$\lim_{q\to-\infty} \int \phi_\lambda \mathrm{d}\nu + q\left[\int f \mathrm{d}\nu - \beta\right] + h(\nu) = \lim_{q\to-\infty} \mathscr{P}(\phi_\lambda + q(f-\beta)) = \infty \tag{4.6.7}$$

by the variational principle and Inequality (4.6.6).

Now, we prove Inequality (4.6.2). To do this, we use our analysis of the limiting behaviour of $q \mapsto \mathscr{P}(\phi_\lambda + q(f-\beta))$. There exists a value $0 \le q^* < q_1 < \infty$ such that

$$\min_{q\ge 0} \mathscr{P}(\phi_\lambda + q(f-\beta)) = \mathscr{P}(\phi_\lambda + q^*(f-\beta)) \le \mathscr{P}(\phi_\lambda)$$

by Equations (4.6.5) and (4.6.7) because pressure is convex. Note that q^* might not be unique. ■

We will also need finite state shifts to compactly approximate pressure. Consider the shift space $\Sigma_{A,n} := \Sigma_A \cap \{1,\dots,n\}^{\mathbb{N}}$. The following lemma will determine our choice of n.

Lemma 4.6.2. *For sufficiently small $\bar{\delta} > \bar{\varepsilon} > 0$ and all $q \in (0,q_1)$, there exist $n \ge N$ such that*

$$\mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q(f-\beta)) > \mathscr{P}(\phi_\lambda + q(f-\beta)) - \bar{\varepsilon} \textit{ and} \tag{4.6.8}$$

$$\mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda) > \mathscr{P}(\phi_\lambda) - \bar{\delta}. \tag{4.6.9}$$

Proof. Fix any $q \in (0,q_1)$.

Consider the finite state shift space $\Sigma_{A,n} := \Sigma_A \cap \{1,\dots,n\}^{\mathbb{N}}$ for each $n \in \mathbb{N}$. Because Σ_A is topologically mixing,

$$\mathscr{P}(\phi_\lambda + q(f-\beta)) = \mathscr{P}_G(\phi_\lambda + q(f-\beta)). \tag{4.6.10}$$

Because of the compact approximation of pressure (see Proposition 2.3.26) and Gurevich and variational pressure are equal (see Equation (4.6.10)),

$$\lim_{n\to\infty} \mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q(f-\beta)) = \mathscr{P}(\phi_\lambda + q(f-\beta)) \text{ and } \lim_{n\to\infty} \mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda) = \mathscr{P}(\phi_\lambda). \tag{4.6.11}$$

Assume that $\bar{\varepsilon} > 0$ is sufficiently small. Consider the set

$$C_n := \left\{q \in (0,q_1) : \mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q(f-\beta)) \le \mathscr{P}(\phi_\lambda + q(f-\beta)) - \bar{\varepsilon}\right\}.$$

Let

$$C := \bigcap_{n=1}^{\infty} C_n.$$

We find that

$$C = \emptyset$$

by Equation (4.6.11).

Therefore, for sufficiently small $\bar{\delta} > \bar{\varepsilon} > 0$ and all $q \in (0, q_1)$, there exist $n \geq N$ such that

$$\mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q(f-\beta)) \geq \mathscr{P}(\phi_\lambda + q(f-\beta)) - \bar{\varepsilon} \text{ and} \tag{4.6.12}$$

$$\mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda) \geq \mathscr{P}(\phi_\lambda) - \bar{\delta}. \tag{4.6.13}$$

■

Take sufficiently small $\bar{\delta} > \bar{\varepsilon} > 0$. By using Inequalities (4.6.8) and (4.6.9) on the compact approximation of pressure, fix an $n \geq N$. Proposition 4.6.1, on the existence of a minimum for pressure, and Lemma 4.6.2, on the compact approximation of pressure, are key elements of the following proposition's proof. This result states an upper bound, a conditional variational principle, for $\min_{q \geq 0} \mathscr{P}(\phi_\lambda + q(f-\beta))$.

Proposition 4.6.3. *Let $\phi_\lambda := -\log|T_\lambda' \circ \pi|$. Take $\bar{N} := (N, N, N, \ldots) \in \Sigma_A$ for each $N \in \mathbb{N}$. Assume that $f : \Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty, \infty)$. There exists a $q^* \in [0, q_1)$ such that*

$$\mathscr{P}(\phi_\lambda + q^*(f-\beta)) = \min_{q \geq 0} \mathscr{P}(\phi_\lambda + q(f-\beta))$$

$$\leq \sup_{\mu \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, \mathrm{d}\mu + h(\mu) : \int f \, \mathrm{d}\mu \geq \beta \right\}. \tag{4.6.14}$$

Proof. In Proposition 4.6.1, we proved that there exists a $q^* \in [0, q_1)$, such that

$$\mathscr{P}(\phi_\lambda + q^*(f-\beta)) = \min_{q \geq 0} \mathscr{P}(\phi_\lambda + q(f-\beta)). \tag{4.6.15}$$

In this proposition's proof, we will use the variational principle, Lemma 4.6.2 on the compact appoximation of pressure, and the convex behaviour of pressure. These results will give us a conditional variational principle that bounds $\mathscr{P}(\phi_\lambda + q^*(f-\beta))$ above.

Fix sufficiently small $\bar{\delta} > \bar{\varepsilon} > 0$, consider the value $q^* \in [0, q_1)$ from Equation (4.6.15), and take an $n \geq N$ according to Inequalities (4.6.8) and (4.6.9). This choice of n will let us use the compact approximation of pressure. Consider the finite state shift $\Sigma_{A,n} := \Sigma_A \cap \{1, \ldots, n\}^{\mathbb{N}}$. Consider the value $q_n^* \in [0, q_1)$ such that

$$\mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q_n^*(f-\beta)) = \min_{q \geq 0} \mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q(f-\beta)). \tag{4.6.16}$$

There exists an equilibrium state $\mu_{q_n^*}$ for $\phi_\lambda + q_n^*(f-\beta)$ by Proposition 2.3.8. The pressure function $q \mapsto \mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q(f-\beta))$ is analytic by Theorem 2.3.10. By Theorem 2.3.10,

$$\frac{\mathrm{d}}{\mathrm{d}q} \mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q(f-\beta))|_{q=q_n^*} = \int f \, \mathrm{d}\mu_{q_n^*} - \beta = 0. \tag{4.6.17}$$

Then,

$$\mathscr{P}_{\Sigma_{A,n}}(\phi_\lambda + q_n^*(f-\beta)) = \sup_{\mu\in M_\sigma(\Sigma_{A,n})}\left\{\int \phi_\lambda \,\mathrm{d}\mu + q_n^*\left[\int f\,\mathrm{d}\mu - \beta\right] + h(\mu) : \int f\,\mathrm{d}\mu = \beta\right\}$$

$$\le \sup_{\mu\in M_\sigma(\Sigma_{A,n})}\left\{\int \phi_\lambda \,\mathrm{d}\mu + h(\mu) : \int f\,\mathrm{d}\mu \ge \beta\right\}. \tag{4.6.18}$$

Let

$$I_n(\beta) := \sup_{\mu\in M_\sigma(\Sigma_{A,n})}\left\{\int \phi_\lambda \,\mathrm{d}\mu + h(\mu) : \int f\,\mathrm{d}\mu \ge \beta\right\}.$$

Hence, because of Proposition 2.3.26, Inequality (4.6.18), and $\{I_n(\beta)\}$ is monotonically increasing, the value $0 \le q^* < q_1$ satisfies

$$\min_{q\ge 0}\mathscr{P}(\phi_\lambda + q(f-\beta)) = \mathscr{P}(\phi_\lambda + q^*(f-\beta)) \le \sup_{\mu\in M_\sigma(\Sigma_A)}\left\{\int \phi_\lambda \mathrm{d}\mu + h(\mu) : \int f\,\mathrm{d}\mu \ge \beta\right\}. \tag{4.6.19}$$

■

Now, we will use this section's results on pressure and the constructed set of cylinders containing periodic points $Per_{\lfloor n\frac{j(k)}{r}\rfloor + M,c}(\beta,1)$ from Section 4.5 to help us form our upper bound (see Theorem 4.7.6) for

$$\limsup_{n\to\infty}\frac{1}{n}\log m(X_\alpha^n).$$

4.7 The Upper Bound for Our Large Deviation Principle

Fix $\alpha \in (L, \alpha_{\sup})$. The results, on periodic points and Gurevich pressure, from the previous subsections will be key to our deviation principle's proof (see Theorem 4.7.2). Consider the set

$$X_\alpha^n := \left\{x = (x_1,\ldots,x_{n-1},x_n,x_{n+1},\ldots)\in\Sigma_A : \frac{\sum_{i=0}^{n-1} f(\sigma^i(x))}{n} \ge \alpha\right\}$$

for an arbitrarily large $n \in \mathbb{N}$. Subsets of X_α^n will help us form an upper bound for

$$\limsup_{n\to\infty}\frac{1}{n}\log m(X_\alpha^n).$$

We revisit the construction of a class of sets $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \subset X_\alpha^n$. First, we will find an upper bound (see Proposition 4.7.1) for $m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right)$. This result will help us bound the lim sup above.

Recall our construction on Section 4.4. Let $\alpha - L > \varepsilon > 0$. Consider the 1−cylinder $[J]$ for each $J \in \mathbb{N}$. Take any $\widehat{J} := (J, x_2, x_3, \ldots) \in [J]$. Recall that Inequality (4.4.2) states: there exists a $M := M(\varepsilon)$ such that for all $J \ge M$,

$$f(\widehat{J}) \le L + \varepsilon. \tag{4.7.1}$$

Take a value $M \in \mathbb{N}$ that satisfies Inequality (4.7.1). We need to recall various subsets of X_α^n for the proof of Theorem 4.7.6.

Now, we restate the constructed subset $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$ from Section 4.4. This use of this set is motivated by our expression for X_α^n (see Proposition 4.4.1 and Equation (4.4.14)). Fix an arbitrarily large $r \in \mathbb{N}$. Take the subset of $[0,1]$ (see Equation (4.4.3)):

$$K_r := \left\{\frac{1}{r}, \frac{2}{r}, \dots, \frac{r-2}{r}, \frac{r-1}{r}, 1\right\}. \tag{4.7.2}$$

Choose an arbitrary $k \in \{1, \dots, r-1, r\}$. For each $j \in (k-1, k]$, take the sets

$$\widehat{X}_M^{n,\lfloor n\frac{j}{r}\rfloor} := \left\{x \in \Sigma_A : x_{\lfloor n\frac{j}{r}\rfloor} \leq M \text{ and } x_i \geq M+1 \ \forall i \in \left[\left\lfloor n\frac{j}{r}\right\rfloor + 1, n\right]\right\}$$

$$\text{and } \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor} := \widehat{X}_M^{n,\lfloor n\frac{j}{r}\rfloor} \cap X_\alpha^n.$$

We take an arbitrary $j(k) \in (k-1, k]$. Hence, we will bound $m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor})$ above to help us find an upper bound for $m(X_\alpha^n)$ (see Proposition 4.7.1).

To form our large deviation principle (see Theorem 4.1.6), we need to consider the following conditional variational principle. For simplicity, we define the function I as

$$I(\gamma) := \sup_{\mu \in M_\sigma(\Sigma_A)} \left\{\int \phi_\lambda \mathrm{d}\mu + h(\mu) : \int f \mathrm{d}\mu \geq \gamma\right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$. To prove our large deviation principle's upper bound (see Theorem 4.7.6), we will form a sequence of propositions, a theorem, and a lemma.

1. First, we will use $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \subset X_\alpha^n$. In particular, we will find an upper bound for

$$m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right)$$

in Proposition 4.7.1. We will use results on pressure (see Section 4.6) to form that bound.

2. Because $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \subset X_\alpha^n$, we will use an expression for X_α^n (see Proposition 4.4.1) and the upper bound for the $\limsup$ (see Proposition 4.7.1). These will help us bound $m(X_\alpha^n)$ above (see Theorem 4.7.2).

3. In Lemma 4.7.3, we will consider a key function β given by

$$\beta(p, \alpha) = \frac{\alpha - (1-p)L}{p}$$

for each $p \in (0,1]$. This function will help us form the weight function for our weighted conditional variational principle and it stems from our expression for α (see Proposition 4.4.2).

4. Then, we prove that $\widetilde{J} : p \mapsto pI(\beta(p,\alpha))$ is uniformly continuous on a compact interval. We reintroduce ε dependence because our subset $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$ for X_α^n and its cover $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$ depend on ε (see Proposition 4.4.2). Consider

$$\beta(p,\alpha,\varepsilon,k,r) = \frac{\alpha-(1-p)(L+\varepsilon)}{p}.$$

   We will approximate the values of the function $\widetilde{J}$ with

$$k \mapsto \frac{k}{r} I(\beta(\alpha,\varepsilon,k,r))$$

   in the proof for Theorem 4.7.6.

5. Finally, by combining our bound for $m(X_\alpha^n)$ from Theorem 4.7.2 and our continuity result for $\widetilde{J}$, we will find the upper bound for

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n)$$

   in Theorem 4.7.6.

Now, we proceed with the first proposition. We find an upper bound for

$$\limsup_{n\to\infty} \frac{1}{n} \log m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right).$$

Proposition 4.7.1. *Assume that $f:\Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty,\infty)$. Recall that*

$$L := \lim_{N\to\infty} f(\bar{N}) \textit{ and } \alpha_{\sup} := \sup_{\nu\in M_\sigma(\Sigma_A)} \left\{\int f \,\mathrm{d}\nu\right\}.$$

Fix $\alpha \in (L,\alpha_{\sup})$. Take any $r\in\mathbb{N}$ and choose an arbitrary $k \in \{1,\dots,r-1,r\}$. For an arbitrary $j(k) \in (k-1,k]$, recall the set $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$ given by Equation (4.4.12)*. Then, there exists a $\beta(\alpha,\varepsilon,j(k),r) \in (\alpha,\alpha_{\sup})$ (given by Proposition 4.4.2) and a constant $D>1$ such that*

$$m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right) \leq \frac{(D)\exp((\lfloor n\frac{j(k)}{r}\rfloor + M+1)I(\beta(\alpha,\varepsilon,j(k),r))}{(\lambda(1-\lambda))^{M+1}}.$$

Proof. Take $\alpha \in (L,\alpha_{\sup})$. Recall our construction from Section 4.4. Take any $r\in\mathbb{N}$ and choose an arbitrary $k\in\{1,\dots,r-1,r\}$. Take an arbitrary $j(k)\in(k-1,k]$. In Proposition 4.4.2, we formed a cover $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$ for $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$. Then, we constructed periodic points from $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}$ in Section 4.5 and proved results on pressure in Section 4.6. Hence, we outline the steps of this proof, using those results, to bound $m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right)$ above. In turn, this upper bound will give us our desired result for the $\limsup$.

1. We use measure theoretic estimates on the sets of cylinders $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}$ and $\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}$ (which were built from $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$).
2. We constructed the set of cylinders $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$ from $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$. Hence, we use the periodic points in $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)$ to bound $m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor})$ above.
3. We get a sum, related to Gurevich pressure, over these points.
4. Then, we use Section 4.6's results on pressure to form an upper bound for $m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right)$.

Now, we will bound $m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor})$ by a sum over periodic points. By Propositions 4.5.1-4.5.3, we use our various sets to form this bound:

$$m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}) \leq m(\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}) = \sum_{\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,c}} m([x_1,\dots,x_{\lfloor n\frac{j(k)}{r}\rfloor}])$$

$$< \left[\frac{1}{\lambda(1-\lambda)}\right]^M \sum_{\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,d}} m([x_1,..,x_{\lfloor n\frac{j(k)}{r}\rfloor},\dots,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1])$$

$$< \left[\frac{1}{\lambda(1-\lambda)}\right]^{M+1} \sum_{Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)} m([1,x_1,..,x_{\lfloor n\frac{j(k)}{r}\rfloor},\dots,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]). \tag{4.7.3}$$

Take the value $\beta := \beta(\alpha,\varepsilon,j(k),r)$ given by Proposition 4.4.2. Recall that there exists a value $q^* \geq 0$ such that

$$\min_{q\geq 0}\mathscr{P}(\phi_\lambda + q(f-\beta)) = \mathscr{P}(\phi_\lambda + q^*(f-\beta)).$$

Then, Proposition 4.2.2, Inequality (4.7.3), and Proposition 4.5.2 give us that

$$m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}) < \left[\frac{1}{\lambda(1-\lambda)}\right]^{M+1} \sum_{Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)} m([1,x_1,..,x_{\lfloor n\frac{j(k)}{r}\rfloor},\dots,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1])$$

$$< \frac{1}{(\lambda(1-\lambda))^{M+1}} \sum_{Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)} \max_{y\in[1,x_1,..,x_{\lfloor n\frac{j(k)}{r}\rfloor},\dots,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]} \exp(S_{\lfloor n\frac{j(k)}{r}\rfloor+M+1}\phi_\lambda(y))$$

$$\leq \frac{D}{(\lambda(1-\lambda))^{M+1}} \sum_{Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(\beta,1)} \max_{y\in[1,x_1,..,x_{\lfloor n\frac{j(k)}{r}\rfloor},\dots,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]} \exp(S_{\lfloor n\frac{j(k)}{r}\rfloor+M+1}(\phi_\lambda + q^*(f-\beta))(y))$$

$$\leq \frac{D}{(\lambda(1-\lambda))^{M+1}} \sum_{Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(1)} \max_{y\in[1,x_1,..,x_{\lfloor n\frac{j(k)}{r}\rfloor},\dots,x_{\lfloor n\frac{j(k)}{r}\rfloor+M-1},1]} \exp(S_{\lfloor n\frac{j(k)}{r}\rfloor+M+1}(\phi_\lambda + q^*(f-\beta))(y))$$

$$\leq \frac{D}{(\lambda(1-\lambda))^{M+1}} \sum_{Per_{\lfloor n\frac{j(k)}{r}\rfloor+M}(1)} \exp(S_{\lfloor n\frac{j(k)}{r}\rfloor+M+1}(\phi_\lambda + q^*(f-\beta))(y)) \tag{4.7.4}$$

for a constant $D>1$ because $\phi_\lambda + q^*(f-\beta)$ is locally Hölder and each cylinder in $Per_{\lfloor n\frac{j(k)}{r}\rfloor+M,c}(1)$ contains a unique $(\lfloor n\frac{j(k)}{r}\rfloor+M)$−periodic point. We note that $D\in(1,\infty)$ because it is a distortion constant and our potentials are locally Hölder.

Now, we will relate Inequality (4.7.4) to Gurevich pressure. By definition of Gurevich pressure, our arbitrarily large n satisfies

(4.7.5)
$$\left|\sum_{Per_{\lfloor n\frac{j(k)}{r}\rfloor+M}(1)} \exp(S_{\lfloor n\frac{j(k)}{r}\rfloor+M+1}(\phi_\lambda+q^*(f-\beta))(y)) - \exp((\lfloor n\frac{j(k)}{r}\rfloor+M+1)\mathscr{P}_G(\phi_\lambda+q^*(f-\beta)))\right| \le \rho(n)$$

for a sufficiently small $\rho(n)\ge 0$.

Hence, by Inequalities (4.7.5) and (4.7.4),

$$m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}) < \frac{D}{(\lambda(1-\lambda))^{M+1}} \sum_{Per_{\lfloor n\frac{j(k)}{r}\rfloor+M}(1)} \exp(S_{\lfloor n\frac{j(k)}{r}\rfloor+M+1}(\phi_\lambda+q^*(f-\beta))(y))$$

$$\le \frac{(D)(\exp((\lfloor n\frac{j(k)}{r}\rfloor+M+1)\mathscr{P}_G(\phi_\lambda+q^*(f-\beta))))}{(\lambda(1-\lambda))^{M+1}} \tag{4.7.6}$$

because we can adjust the value of the distortion constant $D>1$.

Now, we will revisit our upper bound for pressure from Proposition 4.6.3. Denote

$$I(\gamma) := \sup_{\mu\in M_\sigma(\Sigma_A)} \left\{\int \phi_\lambda \mathrm{d}\mu + h(\mu) : \int f \mathrm{d}\mu \ge \gamma\right\}$$

for each $\gamma\in(\alpha,\alpha_{\sup})$. Recall that I gives us an upper bound for pressure (see Proposition 4.6.3).

Therefore,

$$m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right) \le \frac{(D)(\exp((\lfloor n\frac{j(k)}{r}\rfloor+M+1)\mathscr{P}_G(\phi_\lambda+q^*(f-\beta))))}{(\lambda(1-\lambda))^{M+1}}$$

$$\le \frac{(D)(\exp((\lfloor n\frac{j(k)}{r}\rfloor+M+1)I(\beta(\alpha,\varepsilon,j(k),r)))}{(\lambda(1-\lambda))^{M+1}} \tag{4.7.7}$$

by Inequality (4.7.6) and Proposition 4.6.3. ■

Because Proposition 4.7.1 provides an upper bound for $m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}\right)$, we will soon bound $m(X_\alpha^n)$ above. Recall the sets

$$X_\alpha^n := \left\{x=(x_1,\dots,x_{n-1},x_n,x_{n+1},\dots)\in\Sigma_A : \frac{\sum_{i=0}^{n-1} f(\sigma^i(x))}{n} \ge \alpha\right\}.$$

To bound $m(X_\alpha^n)$ above, we need to recall different classes of subsets of X_α^n. Recall our construction from Section 4.4. Take any $r\in\mathbb{N}$ and choose an arbitrary $k\in\{1,\dots,r-1,r\}$. We form a subset of $[0,1]$ (see Equation (4.4.3)):

$$K_r := \left\{\frac{1}{r},\frac{2}{r},\dots,\frac{r-2}{r},\frac{r-1}{r},1\right\}. \tag{4.7.8}$$

We chose an $M \in \mathbb{N}$ according to Inequality (4.7.1). Consider the sets

$$X_M^{n,\lfloor n\frac{k}{r}\rfloor} :=$$
$$\left\{x \in \Sigma_A : \exists j \in \left[\left\lfloor n\frac{(k-1)}{r}\right\rfloor + 1, \left\lfloor n\frac{k}{r}\right\rfloor\right] \text{ such that } x_j \le M \text{ and } x_i \ge M+1\ \forall i \in \left[\left\lfloor n\frac{k}{r}\right\rfloor + 1, n\right]\right\}$$
$$\text{and } X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor} := X_M^{n,\lfloor n\frac{k}{r}\rfloor} \cap X_\alpha^n.$$

Proposition 4.4.1 states that

$$X_\alpha^n = \bigcup_{k=1}^{r} X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}. \tag{4.7.9}$$

Then, the set $\bigcup_{k=1}^r X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ helps us form an upper bound for $m(X_\alpha^n)$.

We consider a set $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor} \subset \bigcup_{l=1}^r X_{\alpha,M}^{n,\lfloor n\frac{l}{r}\rfloor}$ because this will help us bound $m\left(\bigcup_{k=1}^r X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}\right)$ above. To form this bound, we consider a class of subsets for $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$. For each $j \in (k-1,k]$, take the sets

$$\widehat{X}_M^{n,\lfloor n\frac{j}{r}\rfloor} := \left\{x \in \Sigma_A : x_{\lfloor n\frac{j}{r}\rfloor} \le M \text{ and } x_i \ge M+1\ \forall i \in \left[\left\lfloor n\frac{j}{r}\right\rfloor + 1, n\right]\right\}$$
$$\text{and } \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor} := \widehat{X}_M^{n,\lfloor n\frac{j}{r}\rfloor} \cap X_\alpha^n.$$

Naturally,

$$X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor} = \cup_{j\in(k-1,k]} \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}. \tag{4.7.10}$$

In the proof for the upper bound of $m(X_\alpha^n)$ (see Theorem 4.7.2), we will use the following values. We gave the motivation for the use of these values in our preceding discussion on the sets $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ and $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}$. Because of our expressions for the sets X_α^n and $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ (respectively see Equations (4.7.9) and (4.7.10)), we consider the following two values to help us bound $m(X_\alpha^n)$ above. Take the value $k_{\max} \in [1,r]$ as follows

$$\max_{1\le k\le r} m\left(X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}\right) = m\left(X_{\alpha,M}^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}\right). \tag{4.7.11}$$

Consider the value $j_{\max} \in (k_{\max}-1, k_{\max}]$ such that

$$\max_{k_{\max}-1\le j\le k_{\max}} m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}\right) = m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right). \tag{4.7.12}$$

Because we constructed periodic points from $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}$, it will be useful to use the upper bound for pressure (see Proposition 4.6.3). Hence, we define the following function:

$$I(\gamma) := \sup_{\mu\in M_\sigma(\Sigma_A)} \left\{\int \phi_\lambda \mathrm{d}\mu + h(\mu) : \int f \mathrm{d}\mu \ge \gamma\right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$. Now, we give the statement for the upper bound for $m(X_\alpha^n)$.

Theorem 4.7.2. *Assume that $f : \Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty,\infty)$. Recall that*

$$L := \lim_{N\to\infty} f(\bar{N}) \text{ and } \alpha_{\sup} := \sup_{\nu\in M_\sigma(\Sigma_A)} \left\{ \int f \,\mathrm{d}\nu \right\}.$$

Fix $\alpha \in (L,\alpha_{\sup})$. Recall our construction on Section 4.4. Take any $r \in \mathbb{N}$. Recall the values $k_{\max} \in [1,r]$ (given by Equation (4.7.11)) and $j_{\max} \in (k_{\max}-1,k_{\max}]$ (defined by Equation (4.7.12)). Consider the set $\widehat{X}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}_{\alpha,M}$ given by Equation (4.4.12). For a sufficiently large $n \in \mathbb{N}$, there exists a $\beta(\alpha,\varepsilon,j_{\max},r) \in (\alpha,\alpha_{\sup})$ (given by Proposition 4.4.2) and constant $D > 1$ such that

$$m(X^n_\alpha) \le \frac{(nD)(\exp((\lfloor n\frac{j_{\max}}{r}\rfloor + M + 1)I(\beta(\alpha,\varepsilon,j_{\max},r)))}{(\lambda(1-\lambda))^{M+1}}.$$

Proof. First, we will bound $m(X^n_\alpha)$ by using its various subsets.

Recall that Proposition 4.4.1 states that

$$X^n_\alpha = \bigcup_{k=1}^{r} X^{n,\lfloor n\frac{k}{r}\rfloor}_{\alpha,M}.$$

Hence, we can bound $m(X^n_\alpha)$ above by using one of the $X^{n,\lfloor n\frac{k}{r}\rfloor}_{\alpha,M}$:

$$m(X^n_\alpha) \le m\left(\bigcup_{k=1}^{r} X^{n,\lfloor n\frac{k}{r}\rfloor}_{\alpha,M}\right) = \sum_{k=1}^{r} m\left(X^{n,\lfloor n\frac{k}{r}\rfloor}_{\alpha,M}\right) \le r \max_{1\le k\le r} m\left(X^{n,\lfloor n\frac{k}{r}\rfloor}_{\alpha,M}\right). \tag{4.7.13}$$

Take the value $k_{\max} \in [1,r]$ as follows:

$$\max_{1\le k\le r} m\left(X^{n,\lfloor n\frac{k}{r}\rfloor}_{\alpha,M}\right) = m\left(X^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}_{\alpha,M}\right). \tag{4.7.14}$$

Then, because $k_{\max}$ is the value that maximises $m\left(X^{n,\lfloor n\frac{k}{r}\rfloor}_{\alpha,M}\right)$,

$$m(X^n_\alpha) \le r m\left(X^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}_{\alpha,M}\right) \tag{4.7.15}$$

by Inequality (4.7.13) and Equation (4.7.14).

Consider the class of subsets for $X^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}_{\alpha,M}$ (as given by Equation (4.4.13)). Take the sets

$$\widehat{X}^{n,\lfloor n\frac{j}{r}\rfloor}_{\alpha,M} := \widehat{X}^{n,\lfloor n\frac{j}{r}\rfloor}_{M} \cap X^n_\alpha$$

for each $j \in (k_{\max}-1,k_{\max}]$. Naturally, we find that

$$X^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}_{\alpha,M} = \bigcup_{j\in(k_{\max}-1,k_{\max}]} \widehat{X}^{n,\lfloor n\frac{j}{r}\rfloor}_{\alpha,M}.$$

by Equation (4.4.14).

Then, we find that the measure of $X_{\alpha,M}^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}$ can be bounded above by using one of its subsets:

(4.7.16)
$$m\left(X_{\alpha,M}^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}\right) \leq \left(\lfloor n\frac{k}{r}\rfloor - \lfloor n\frac{(k-1)}{r}\rfloor\right) \max_{k_{\max}-1\leq j\leq k_{\max}} m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}\right) \leq \frac{n}{r} \max_{k_{\max}-1\leq j\leq k_{\max}} m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}\right)$$

because

$$\lfloor n\frac{k}{r}\rfloor - \lfloor n\frac{(k-1)}{r}\rfloor \leq \frac{n}{r}. \tag{4.7.17}$$

There exists a value $j_{\max} \in (k-1,k]$ such that

$$\max_{k_{\max}-1\leq j\leq k_{\max}} m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}\right) = m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right). \tag{4.7.18}$$

Then,

$$m\left(X_{\alpha,M}^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}\right) \leq \frac{n}{r} m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right) \tag{4.7.19}$$

by Inequality (4.7.16) and Equation (4.7.18).

Whence, we combine the upper bounds for $m(X_\alpha^n)$ (see Inequality (4.7.15)) and $m\left(X_{\alpha,M}^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}\right)$ (see Inequality (4.7.19)):

$$m(X_\alpha^n) \leq rm\left(X_{\alpha,M}^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}\right) \leq r\left[\frac{n}{r} m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right)\right] \leq nm\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right) \tag{4.7.20}$$

Now, we will use our upper bound for

$$m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right)$$

from Proposition 4.7.1. Recall that $\beta = \beta(\alpha,\varepsilon,j_{\max},r)$ is given by Proposition 4.4.2 as follows:

$$\beta(\alpha,\varepsilon,j_{\max},r) = \frac{\alpha - (1-\frac{j_{\max}}{r})(L+\varepsilon)}{\frac{j_{\max}}{r}}.$$

Denote

$$I(\gamma) := \sup_{\mu\in M_\sigma(\Sigma_A)} \left\{\int \phi_\lambda \mathrm{d}\mu + h(\mu) : \int f \mathrm{d}\mu \geq \gamma\right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$.

If $n \in \mathbb{N}$ is sufficiently large, there exists a constant $D > 1$ such that

$$m(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}) \leq \frac{D\exp((\lfloor n\frac{j(k)}{r}\rfloor + M + 1)I(\beta(\alpha,\varepsilon,j_{\max},r))}{(\lambda(1-\lambda))^{M+1}} \tag{4.7.21}$$

by Proposition 4.7.1.

Thus, we find an upper bound:

$$m(X_\alpha^n) \leq nm\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right) \leq \frac{nD\exp((\lfloor n\frac{j_{\max}}{r}\rfloor + M + 1)I(\beta(\alpha,\varepsilon,j_{\max},r))}{(\lambda(1-\lambda))^{M+1}} \tag{4.7.22}$$

by Inequalities (4.7.20) and (4.7.21). ∎

The following lemma will help us use the upper bound for $m(X^n_\alpha)$ (see Theorem 4.7.2) to prove that a weighted conditional variational principle is the upper bound for our large deviation principle (see Theorem 4.7.6). In particular, it will help us use our cover $\widehat{X}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor}_{\beta,M}$ for $\widehat{X}^{n,\lfloor n\frac{j(k)}{r}\rfloor}_{\alpha,M}$.

First, we motivate our use of a function in the following lemma. Take an arbitrary $r \in \mathbb{N}$. Consider an arbitrary $k \in \{1,...,r\}$ and any value $j(k) \in (k,k+1]$. Using k, r, and $j(k)$, recall the constructed subsets of X^n_α from Subsection 4.4. Let $p = \frac{j(k)}{r}$. We considered a subset $\widehat{X}^{n,\lfloor np\rfloor}_{\alpha,M}$ formed of sequences that follow orbits of recurrent points until time $\lfloor np\rfloor$. Let $\beta := \beta(p,\alpha,\varepsilon) = \frac{\alpha-(1-p)(L+\varepsilon)}{p}$ was chosen in Proposition 4.4.2. We constructed a cover $\widehat{X}^{\lfloor np\rfloor,\lfloor np\rfloor}_{\beta,M}$ for $\widehat{X}^{n,\lfloor np\rfloor}_{\alpha,M}$ in Proposition 4.4.2.

Because β depends on various variables, we introduce of the function $\beta : p \mapsto \frac{\alpha-(1-p)L}{p}$. The range of the function β will be taken as $(\alpha,\alpha_{\sup})$ because $\beta(p,\alpha,\varepsilon) \in (\alpha,\alpha_{\sup})$. To define the domain of β, we will need to find the value $p_{\inf}$ such that

$$\frac{\alpha-(1-p_{\inf})L}{p_{\inf}} = \alpha_{\sup}.$$

Lemma 4.7.3. *Fix an* $\alpha \in (L,\alpha_{\sup})$. *Consider the function* $\beta : p \mapsto \frac{\alpha-(1-p)L}{p}$ *on* $(0,1]$. *This is a decreasing function. Furthermore, there exists a value*

$$p_{\inf} := \frac{\alpha-L}{\alpha_{\sup}-L} > 0$$

such that

$$\frac{\alpha-(1-p_{\inf})L}{p_{\inf}} = \alpha_{\sup}.$$

Proof. We find that $\beta'(p) < 0$ for each $p \in (0,1)$ because $L < \alpha$. Hence, β is a decreasing, continuous function.

Without loss of generality, assume that $\alpha_{\sup} < \infty$. We find that

$$\lim_{p\to 0^+}\beta(p) = \lim_{p\to 0^+}\frac{\alpha-L}{p} + L = \infty. \tag{4.7.23}$$

Because β is divergent (see Equation (4.7.23)) and continuous (as $\beta(p) = \frac{\alpha-L}{p}+L$), there exists a value $p_{\inf} \in (0,1)$ such that

$$\alpha_{\sup} = \beta(p_{\inf}) = \frac{\alpha-(1-p_{\inf})L}{p_{\inf}}. \tag{4.7.24}$$

Because β is decreasing, $\beta(p) > \alpha_{\sup}$ for any $p < p_{\inf}$. Therefore, there exists a value

$$p_{\inf} := \inf\left\{p \in (0,1) : \frac{\alpha-(1-p)L}{p} \in (\alpha,\alpha_{\sup})\right\} = \left\{p \in (0,1) : \frac{\alpha-(1-p)L}{p} = \alpha_{\sup}\right\} = \frac{\alpha-L}{\alpha_{\sup}-L}.$$

∎

To form a weighted conditional variational principle as our upper bound (see Theorem 4.7.6), we recall the definition of the function I :

$$I(\beta) := \sup_{\mu\in M_\sigma(\Sigma_A)}\left\{\int \phi_\lambda \,\mathrm{d}\mu + h(\mu) : \int f \,\mathrm{d}\mu \geq \beta\right\} \tag{4.7.25}$$

for each $\beta \in (\alpha, \alpha_{\sup})$. We previously saw this function in our upper bound for pressure (see Proposition 4.6.3). The following lemma proves that $\beta \mapsto I(\beta)$ is uniformly continuous on a closed interval. We will prove that I is concave because this will be the key to proving its continuity.

Lemma 4.7.4. *Assume that $f : \Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty, \infty)$. Fix an $\alpha \in (L, \alpha_{\sup})$. The function I is a uniformly continuous, concave, and bounded function on $[\alpha, \alpha_{\sup}]$.*

Proof. We will first prove that I is concave. Then, we will prove that I is uniformly continuous. Consider the interval $(\alpha, \alpha_{\sup})$. Take two arbitrary values $\alpha < \beta_1 < \beta_2 < \alpha_{\sup}$ and consider any $p \in (0,1)$.

To prove that I is concave, we construct the following two measures. Take a sufficiently small $\varepsilon_1 > 0$. Consider a measure $\mu_1 \in M_\sigma(\Sigma_A)$ such that

$$I(\beta_1) - \left[\int \phi_\lambda \mathrm{d}\mu_1 + h(\mu_1)\right] \le \varepsilon_1 \text{ and} \tag{4.7.26}$$

$$\int f \, \mathrm{d}\mu_1 \ge \beta_1. \tag{4.7.27}$$

Take a sufficiently small $\varepsilon_2 > 0$. Consider a measure $\mu_2 \in M_\sigma(\Sigma_A)$ such that

$$I(\beta_2) - \left[\int \phi_\lambda \mathrm{d}\mu_2 + h(\mu_2)\right] \le \varepsilon_2 \text{ and} \tag{4.7.28}$$

$$\int f \, \mathrm{d}\mu_2 \ge \beta_2. \tag{4.7.29}$$

To prove concavity, we take the convex combination of μ_1 and μ_2. For an arbitrary $p \in (0,1)$, take a measure $\mu \in M_\sigma(\Sigma_A)$ such that

$$\mu := p\mu_2 + (1-p)\mu_1. \tag{4.7.30}$$

Hence,

$$\int f \, \mathrm{d}\mu \ge p\beta_2 + (1-p)\beta_1 \tag{4.7.31}$$

by Equation (4.7.30) and Inequalities (4.7.27) and (4.7.29).

We will show that I satisfies the condition for concavity:

$$I(p\beta_2 + (1-p)\beta_1) \ge pI(\beta_2) + (1-p)I(\beta_1)$$

for any $\alpha < \beta_1 < \beta_2 < \alpha_{\sup}$.

We find that

$$(4.7.32)\quad I(p\beta_2+(1-p)\beta_1)=\sup_{\nu\in M_\sigma(\Sigma_A)}\left\{\int\phi_\lambda\,\mathrm{d}\nu+h(\nu):\int f\,\mathrm{d}\nu\geq p\beta_2+(1-p)\beta_1\right\}\geq\int\phi_\lambda\,\mathrm{d}\mu+h(\mu)$$

by Inequality (4.7.31).

Furthermore,

$$(4.7.33)\quad \int\phi_\lambda\,\mathrm{d}\mu+h(\mu)=\int\phi_\lambda\,\mathrm{d}[p\mu_2+(1-p)\mu_1]+h(p\mu_2+(1-p)\mu_1)\geq p\int\phi_\lambda\,\mathrm{d}\mu_2+(1-p)\int\phi_\lambda\,\mathrm{d}\mu_1+ph(\mu_2)+(1-p)h(\mu_1)$$

because the entropy function is concave.

Hence,

$$(4.7.34)\quad \begin{aligned}\int\phi_\lambda\,\mathrm{d}\mu+h(\mu)&\geq p\left[\int\phi_\lambda\,\mathrm{d}\mu_2+h(\mu_2)\right]+(1-p)\left[\int\phi_\lambda\,\mathrm{d}\mu_1+h(\mu_1)\right]\\&\geq pI(\beta_2)-p\varepsilon_2+(1-p)I(\beta_1)-(1-p)\varepsilon_1\end{aligned}$$

by Equation (4.7.30) and Inequalities (4.7.26) and (4.7.28).

Therefore, I is concave, i.e.,

$$(4.7.35)\quad I(p\beta_2+(1-p)\beta_1)\geq pI(\beta_2)+(1-p)I(\beta_1)$$

because of Inequalities (4.7.32), (4.7.33), and (4.7.34) and $\varepsilon_1,\varepsilon_2>0$ are arbitrarily small.

Because I is concave, this function is continuous on $(\alpha,\alpha_{\text{sup}})$. Hence, to prove that I is uniformly continuous on $[\alpha,\alpha_{\text{sup}}]$, we need to show that I is a bounded and decreasing function.

First, we will prove that I is a bounded function. Equation (4.7.25) defines that function:

$$I(\beta):=\sup_{\mu\in M_\sigma(\Sigma_A)}\left\{\int\phi_\lambda\mathrm{d}\mu+h(\mu):\int f\,\mathrm{d}\mu\geq\beta\right\}$$

for each $\beta\in(\alpha,\alpha_{\text{sup}})$. First, we show that $I(\beta)>-\infty$. By definition, $\phi_\lambda(x)\geq\log[\lambda(1-\lambda)]$ for each $x\in\Sigma_A$ and $h(\mu)\geq 0$ for any $\mu\in M_\sigma(\Sigma_A)$. Hence,

$$(4.7.36)\quad I(\beta)\geq\log[\lambda(1-\lambda)]>-\infty.$$

Furthermore,

$$(4.7.37)\quad I(\beta)\leq\mathscr{P}(\phi_\lambda)<0$$

by the variational principle. Thus,

$$I(\beta)\in(-\infty,0)$$

and hence, I is a bounded function on the closed interval $[\alpha,\alpha_{\text{sup}}]$.

Now, we prove that I is a decreasing function. Let us take two arbitrary values $\alpha\leq\beta_1<\beta_2\leq\alpha_{\text{sup}}$. We find that

$$(4.7.38)\quad \sup_{\mu\in M_\sigma(\Sigma_A)}\left\{\int\phi_\lambda\mathrm{d}\mu+h(\mu):\int f\,\mathrm{d}\mu\geq\beta_2\right\}\leq\sup_{\mu\in M_\sigma(\Sigma_A)}\left\{\int\phi_\lambda\mathrm{d}\mu+h(\mu):\int f\,\mathrm{d}\mu\geq\beta_1\right\}.$$

This gives us that $I(\beta_2) \le I(\beta_1)$, so I is a decreasing function.

Therefore, I is a uniformly continuous function on $[\alpha, \alpha_{\sup}]$ because it is continuous on $(\alpha, \alpha_{\sup})$ and decreasing and bounded on $[\alpha, \alpha_{\sup}]$. ■

Now, we build a function from I. We will consider the function

$$\widetilde{J} : p \mapsto pI(\beta(p,\alpha)) \tag{4.7.39}$$

because its values will lead to our weighted conditional variational principle (see Theorem 4.7.6). When $p = \frac{j_{\max}}{r}$, we found that $I(\beta(\alpha, \varepsilon, p))$ is an upper bound for pressure (see Proposition 4.6.3). In Proposition 4.7.1, we proved that $m(X_\alpha^n)$ is bounded above by the exponential function of $I(\beta(\alpha,\varepsilon,p))$ (see Equation (4.7.11)). Hence, to proceed, we will prove that $\widetilde{J}$ is uniformly continuous.

Proposition 4.7.5. *Assume that $f : \Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty,\infty)$. Fix an $\alpha \in (L, \alpha_{\sup})$. Consider the value*

$$p_{\inf} := \frac{\alpha - L}{\alpha_{\sup} - L}.$$

For each $p \in [p_{\inf}, 1]$, take the value

$$\beta(p,\alpha) = \frac{\alpha - (1-p)L}{p}.$$

The function

$$\widetilde{J} : p \mapsto pI(\beta(p,\alpha))$$

is a bounded, uniformly continuous function on $[p_{\inf}, 1]$.

Proof. The keys to proving that

$$\widetilde{J} : p \mapsto pI(\beta(p,\alpha))$$

is a bounded, uniformly continuous function on $[p_{\inf}, 1]$ are the behaviour of the function I (see Lemma 4.7.4) and the composition of the two functions I and β. The function β given by

$$\beta(p,\alpha) = \frac{\alpha - (1-p)L}{p} \tag{4.7.40}$$

is continuous. By Lemma 4.7.4, the function $I : \beta \mapsto I(\beta)$ is uniformly continuous and bounded on $[\alpha, \alpha_{\sup}]$. Note that $\beta^{-1}([\alpha, \alpha_{\sup}]) = [p_{\inf}, 1]$ by Lemma 4.7.3. Hence,

$$p \mapsto I(\beta(p,\alpha))$$

is uniformly continuous on $[p_{\inf}, 1]$.

It naturally follows that $\widetilde{J} : p \mapsto pI(\beta(p,\alpha))$ is bounded and uniformly continuous on $[p_{\inf}, 1]$ because of Lemma 4.7.4 and $p \mapsto I(\beta(p,\alpha))$ is uniformly continuous on $[p_{\inf}, 1]$. ■

Finally, we will use Theorem 4.7.2, Lemma 4.7.3, and Proposition 4.7.5 to form our upper bound (see Theorem 4.7.6) because these results give us a necessary upper bound for $m(X_\alpha^n)$ and state that $\widetilde{J}: p \mapsto pI(\beta(p,\alpha))$ is continuous. We will form an approximation argument (see Inequality (4.7.59)), which uses a function built from $\widetilde{J}$, to obtain our weighted conditional variational principle in Theorem 4.7.6.

4.7.1 Statement and Proof of the Upper Bound

Fix $\alpha \in (L, \alpha_{\sup})$. We will an upper bound for $m(X_\alpha^n)$ (see Theorem 4.7.2) and Section 4.6, which states a result that bounds pressure above, to form our upper bound for our large deviation principle (see Theorem 4.7.6). Fix an arbitrarily large $n \in \mathbb{N}$. Recall the set

$$X_\alpha^n := \left\{ x = (x_1, ..., x_{n-1}, x_n, x_{n+1}, ...) \in \Sigma_A : \frac{\sum_{i=0}^{n-1} f(\sigma^i(x))}{n} \geq \alpha \right\}.$$

We will soon form an upper bound (see Theorem 4.7.6) for

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n).$$

To do this, we need to recall various subsets of X_α^n for the proof of Theorem 4.7.6. We used these subsets to bound $m(X_\alpha^n)$ above (see Theorem 4.7.2). These sets were constructed in Subsection 4.4.

Fix an arbitrarily large $r \in \mathbb{N}$. Take the following subset of $[0,1]$ (see Equation (4.4.3)):

$$K_r := \left\{ \frac{1}{r}, \frac{2}{r}, ..., \frac{r-2}{r}, \frac{r-1}{r}, 1 \right\}. \tag{4.7.41}$$

We chose an $M \in \mathbb{N}$ according to Inequality (4.7.1). Consider an arbitrary $k \in \{1, ..., r-1, r\}$.

Consider the sets

$$X_M^{n, \lfloor n\frac{k}{r} \rfloor} :=$$

$$\left\{ x \in \Sigma_A : \exists j \in \left[\left\lfloor n\frac{(k-1)}{r} \right\rfloor + 1, \left\lfloor n\frac{k}{r} \right\rfloor \right] \text{ such that } x_j \leq M \text{ and } x_i \geq M+1 \; \forall i \in \left[\left\lfloor n\frac{k}{r} \right\rfloor + 1, n \right] \right\} \tag{4.7.42}$$

$$\text{and } X_{\alpha,M}^{n, \lfloor n\frac{k}{r} \rfloor} := X_M^{n, \lfloor n\frac{k}{r} \rfloor} \cap X_\alpha^n. \tag{4.7.43}$$

Recall that we found an expression for X_α^n (see Proposition 4.4.1):

$$X_\alpha^n = \bigcup_{k=1}^{r} X_{\alpha,M}^{n, \lfloor n\frac{k}{r} \rfloor}.$$

Hence, we take a set $X_{\alpha,M}^{n, \lfloor n\frac{k}{r} \rfloor}$. We consider a classes of subsets for this set. For each $j \in (k-1, k]$, take the sets

$$\widehat{X}_M^{n, \lfloor n\frac{j}{r} \rfloor} := \left\{ x \in \Sigma_A : x_{\lfloor n\frac{j}{r} \rfloor} \leq M \text{ and } x_i \geq M+1 \; \forall i \in \left[\left\lfloor n\frac{j}{r} \right\rfloor + 1, n \right] \right\} \tag{4.7.44}$$

$$\text{and } \widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor} := \widehat{X}_{M}^{n,\lfloor n\frac{j}{r}\rfloor} \cap X_{\alpha}^{n}. \tag{4.7.45}$$

Naturally, we find that

$$X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor} = \cup_{j\in(k-1,k]}\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}.$$

Consider an arbitrary $j(k) \in (k-1,k]$. We will also need to consider the cover of each $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j(k)}{r}\rfloor}$. For our $\beta \in (\alpha,\alpha_{\sup})$ (see Proposition 4.4.2), define the set

$$\widehat{X}_{\beta,M}^{\lfloor n\frac{j(k)}{r}\rfloor,\lfloor n\frac{j(k)}{r}\rfloor} := \widehat{X}_{M}^{n,\lfloor n\frac{j(k)}{r}\rfloor} \cap X_{\beta}^{\lfloor n\frac{j(k)}{r}\rfloor}.$$

Define the function β as

$$\beta(p,\alpha) = \frac{\alpha-(1-p)L}{p} \tag{4.7.46}$$

for each $p \in (p_{\inf},1]$. In Proposition 4.7.5, we proved that for each $\beta \in (\alpha,\alpha_{\sup})$, there exists a value p such that $\beta = \beta(p,\alpha)$.

Hence, we consider the function $p \to pI(\beta(p,\alpha))$. That function will help us form a weighted conditional variational principle in Theorem 4.7.6. Recall the value

$$p_{\inf} = \frac{\alpha-L}{\alpha_{\sup}-L}$$

from Lemma 4.7.3. Take the values $p(\alpha) \in (p_{\inf},1]$ and $\beta(p(\alpha),\alpha) := \beta(\alpha) \in (\alpha,\alpha_{\sup})$ such that

$$\beta(\alpha) = \frac{\alpha-(1-p(\alpha))L}{p(\alpha)} \tag{4.7.47}$$

and

$$\max_{p_{\inf}\le p\le 1} pI(\beta(p,\alpha)) = p(\alpha)I(\beta(\alpha)). \tag{4.7.48}$$

We will prove that these values form our upper bound for our large deviation principle.

Theorem 4.7.6. *Fix $\lambda \in \left(\frac{1}{2},1\right)$. Recall the map T_λ given by Equation* (4.1.1)*, the coding map $\pi : \Sigma_A \to (0,1]$, and the shift space (Σ_A,σ). Let $\phi_\lambda := -\log|T_\lambda^{'} \circ \pi|$. Take $\bar{N} := (N,N,N,\ldots) \in \Sigma_A$ for each $N \in \mathbb{N}$. Assume that $f : \Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty,\infty)$. Recall that*

$$L := \lim_{N\to\infty} f(\bar{N}) \text{ and } \alpha_{\sup} := \sup_{\nu\in M_\sigma(\Sigma_A)} \left\{\int f\,\mathrm{d}\nu\right\}.$$

Fix $\alpha \in \left(L,\alpha_{\sup}\right)$. Consider the value

$$p_{\inf} := \frac{\alpha-L}{\alpha_{\sup}-L}.$$

Then, there exist a $p(\alpha) \in (p_{\inf},1]$ and $\beta(\alpha) \in (\alpha,\alpha_{\sup})$ (given by Equations (4.7.46), (4.7.47)*, and* (4.7.48)*) such that*

$$\limsup_{n\to\infty} \frac{1}{n}\log m(X_\alpha^n) \le p(\alpha)\left(\sup_{\eta\in M_\sigma(\Sigma_A)}\left\{\int \phi_\lambda\,\mathrm{d}\eta + h(\eta) : \int f\,\mathrm{d}\eta \ge \beta(\alpha)\right\}\right). \tag{4.7.49}$$

Proof. Take $\alpha \in (L, \alpha_{\sup})$. We will prove that a weighted conditional variational principle is our upper bound for

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n).$$

To do this, we will prove that $p(\alpha)$ and $\beta(\alpha)$, from Equations (4.7.47) and (4.7.48), form this bound. To form the weighted conditional variational principle, we need to revisit the function I (see Equation (4.7.25)) and functions built from I. We explain more in the following outline.

1. Consider the value
$$p_{\inf} := \frac{\alpha - L}{\alpha_{\sup} - L}.$$
For each $p \in (p_{\inf}, 1]$, take the value
$$\beta(p,\alpha) = \frac{\alpha - (1-p)L}{p}.$$
We will again consider the function
$$\widetilde{J} : p \mapsto pI(\beta(p,\alpha)).$$

2. We will use Proposition 4.7.5, which states that $\widetilde{J}$ is uniformly continuous. We use $\widetilde{J}$ to define a continuous function $\widetilde{I}$ (see Equation (4.7.51)). We will approximate (see Inequality (4.7.59)) $\widetilde{I}$ with a discrete function. This discrete function comes from our upper bound for $m(X_\alpha^n)$ (see Theorem 4.7.2).

3. Finally, we use the upper bound for $m(X_\alpha^n)$ from Theorem 4.7.2 and the approximation of $\tilde{I}$ (see Inequality (4.7.59)) to prove that there exist a $p(\alpha) \in (p_{\inf}, 1]$ and $\beta(\alpha) \in (\alpha, \alpha_{\sup})$ (given by Equations (4.7.47) and (4.7.48)) such that
$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \le p(\alpha) I(\beta(\alpha)).$$

We proceed by analysing the function $\tilde{I}$. Recall the value

$$p_{\inf} := \frac{\alpha_{\sup} - L}{\alpha - L}.$$

First, let us define the value $\beta(p,\alpha)$. For each $p \in (p_{\inf}, 1]$, define the value $\beta(p,\alpha)$ as

$$\beta(p,\alpha) = \frac{\alpha - (1-p)L}{p}. \tag{4.7.50}$$

Take the function $\widetilde{J} : p \mapsto pI(\beta(p,\alpha))$. Then, consider the function $\tilde{I}$ such that

$$\tilde{I}(\alpha) = \max_{p_{\inf} \le p \le 1} pI(\beta(p,\alpha)). \tag{4.7.51}$$

Our aim will be to show that $\tilde{I}(\alpha)$ bounds the $\limsup$ above (see Inequality (4.7.63)).

Now, let us introduce ε dependence again and consider $\tilde{I}$ when this dependence is present. We will relate $p\widetilde{I}(p,\alpha,\varepsilon)$ (see Equation (4.7.54)) to $p\widetilde{I}(p,\alpha)$ (see Equation (4.7.57)). Take an $\varepsilon > 0$. Define the value $p_{\text{inf}}(\varepsilon) \in (0,1]$ as

$$p_{\text{inf}}(\varepsilon) := \frac{\alpha - L - \varepsilon}{\alpha_{\text{sup}} - L - \varepsilon}. \tag{4.7.52}$$

For each $p \in (p_{\text{inf}}(\varepsilon), 1]$, define the value $\beta(p,\alpha,\varepsilon)$ as

$$\beta(p,\alpha,\varepsilon) = \frac{\alpha - (1-p)(L+\varepsilon)}{p}. \tag{4.7.53}$$

Take the function $p \mapsto pI(\beta(p,\alpha,\varepsilon))$. Then, consider

$$\tilde{I}(\alpha,\varepsilon) = \max_{p_{\text{inf}}(\varepsilon) \le p \le 1} pI(\beta(p,\alpha,\varepsilon)). \tag{4.7.54}$$

We will later consider a discrete version of this function

$$\widetilde{I}_r(\alpha,\varepsilon) = \max_{1 \le k \le r} \frac{k}{r} I(\beta(\alpha,\varepsilon,k,r)).$$

Then, we will approximate the values of $\widetilde{I}$ by using that function (see Inequality (4.7.59)). That step will be key to this proof.

Next, we need to consider the limiting behaviour for $\tilde{I}(\alpha,\varepsilon)$ and $\widetilde{I}_r(\alpha,\varepsilon)$. By definition of $p_{\text{inf}}(\varepsilon)$ (see Equation (4.7.52)),

$$\lim_{\varepsilon \to 0} p_{\text{inf}}(\varepsilon) = p_{\text{inf}}. \tag{4.7.55}$$

Now, we find that

$$\lim_{\varepsilon \to 0} \tilde{I}(\alpha,\varepsilon) = \tilde{I}(\alpha) \tag{4.7.56}$$

because

$$\lim_{\varepsilon \to 0} \max_{p_{\text{inf}}(\varepsilon) \le p \le 1} pI(\beta(p,\alpha,\varepsilon)) = \max_{p_{\text{inf}} \le p \le 1} pI(\beta(p,\alpha)) \tag{4.7.57}$$

by Equations (4.7.54), (4.7.51), and (4.7.55).

We will now construct the function $\widetilde{I}_r$ (i.e., the discrete version of $\widetilde{I}$). In Theorem 4.7.2, we found that $m(X_\alpha^n)$ is bounded above by the exponential function of $I(\beta(\alpha,\varepsilon,j(k),r))$. For simplicity, we will consider the function $k \mapsto \frac{k}{r} I(\beta(\alpha,\varepsilon,k,r))$ and connect it to the function $\widetilde{I}$ and the limit given by Equation (4.7.56). We found that $\beta(\alpha,\varepsilon,k,r)$ was given by Proposition 4.4.2:

$$\beta(\alpha,\varepsilon,k,r) = \frac{\alpha - \left(1 - \frac{k}{r}\right)(L+\varepsilon)}{\frac{k}{r}}.$$

Now, we proceed with our approximation argument. Consider the discrete function $k \mapsto \frac{k}{r} I(\beta(\alpha,\varepsilon,k,r))$ for each $k \in \{1,\dots,r\}$ and take the function $\widetilde{I}_r$ given by

$$\widetilde{I}_r(\alpha,\varepsilon) = \max_{1 \le k \le r} \frac{k}{r} I(\beta(\alpha,\varepsilon,k,r)).$$

By Proposition 4.7.5, $\widetilde{I}: p \to pI(\beta(p,\alpha,\varepsilon))$ on $(p_{\inf}(\varepsilon),1]$ is a continuous function. Because

$$\frac{k}{r} - \frac{k-1}{r} = \frac{1}{r}, \tag{4.7.58}$$

the discrete function $k \mapsto \frac{k}{r} I(\beta(\alpha,\varepsilon,k,r))$ approximates the continuous function $p \to pI(\beta(p,\alpha,\varepsilon))$. Hence, there exists a sufficiently small $\widetilde{\theta}(r,\varepsilon) > 0$ such that

$$|\widetilde{I}_r(\alpha,\varepsilon) - \widetilde{I}(\alpha,\varepsilon)| = \left| \max_{1\le k\le r} \frac{k}{r} I(\beta(\alpha,\varepsilon,k,r)) - \max_{p_{\inf}(\varepsilon)\le p\le 1} pI(\beta(p,\alpha,\varepsilon)) \right| \le \widetilde{\theta}(r,\varepsilon). \tag{4.7.59}$$

We will be able to take limits with respect to ε and r because our partition $K_r := \{\frac{1}{r},\dots,\frac{r-1}{r},1\}$ was defined by our arbitrary $r \in \mathbb{N}$ and $\varepsilon > 0$ (see Inequality (4.4.2)). Thus,

$$\lim_{\varepsilon\to 0}\lim_{r\to\infty}\max_{1\le k\le r} \frac{k}{r} I(\beta(\alpha,\varepsilon,k,r)) = \lim_{\varepsilon\to 0}\max_{p_{\inf}(\varepsilon)\le p\le 1} pI(\beta(p,\alpha,\varepsilon)) = \max_{p_{\inf}\le p\le 1} pI(\beta(p,\alpha)) \tag{4.7.60}$$

by Inequality (4.7.59) and the definition of $p_{\inf}(\varepsilon)$ (see Equation (4.7.52)).

Take a sufficiently large $n \in \mathbb{N}$. To revisit our upper bound for $m(X_\alpha^n)$ (see Theorem 4.7.2), we will recall two subsets and values taken for these sets. Consider the set $X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}$ (see Equations (4.7.42) and (4.7.43)). Because Proposition 4.4.1 states that $\bigcup_{k=1}^r X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor} = X_\alpha^n$,

$$m(X_\alpha^n) = m\left(\bigcup_{k=1}^r X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}\right)$$

Hence, we take the value $k_{\max} := k_{\max}(r) \in [1,r]$ as follows

$$\max_{1\le k\le r} m\left(X_{\alpha,M}^{n,\lfloor n\frac{k}{r}\rfloor}\right) = m\left(X_{\alpha,M}^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}\right). \tag{4.7.61}$$

Now, we express $X_{\alpha,M}^{n,\lfloor n\frac{k_{\max}}{r}\rfloor}$ as a union of subsets. Recall the set $\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}$ (see Equations (4.7.44) and (4.7.45)) for each $j \in (k_{\max}-1, k_{\max}]$. Consider the value $j_{\max} := j_{\max}(r) \in (k_{\max}-1,k_{\max}]$ such that

$$\max_{k_{\max}-1\le j\le k_{\max}} m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j}{r}\rfloor}\right) = m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right). \tag{4.7.62}$$

Hence, we will bound $m(X_\alpha^n)$ above by using the upper bound for $m\left(\widehat{X}_{\alpha,M}^{n,\lfloor n\frac{j_{\max}}{r}\rfloor}\right)$ (see Theorem 4.7.2).

Recall that $\beta = \beta(\alpha,\varepsilon,j_{\max}(r),r)$ is given by Proposition 4.4.2 as follows:

$$\beta(\alpha,\varepsilon,j_{\max}(r),r) = \frac{\alpha - (1-\frac{j_{\max}(r)}{r})(L+\varepsilon)}{\frac{j_{\max}(r)}{r}}.$$

Theorem 4.7.2 states: there exists a constant $D > 1$ such that

$$m(X_\alpha^n) \le \frac{nD\exp((\lfloor n\frac{j_{\max}}{r}\rfloor + M + 1)I(\beta(\alpha,\varepsilon,j_{\max},r))}{(\lambda(1-\lambda))^{M+1}}.$$

Then, we now combine our preceding results about the function $\widetilde{I}$ (see Equation (4.7.60)) and our upper bound for $m(X_\alpha^n)$ (see Theorem 4.7.2). Again, recall the subset of $[0,1]$:

$$K_r := \left\{ \frac{1}{r}, \dots, \frac{r-1}{r}, 1 \right\}.$$

Because $k_{\max}(r) - 1 < j_{\max}(r) \le k_{\max}(r)$ and

$$\frac{k_{\max}(r)}{r} - \frac{k_{\max}(r)-1}{r} = \frac{1}{r},$$

we find that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \le \lim_{\varepsilon\to 0} \lim_{r\to\infty} \frac{k_{\max}(r)}{r} I(\beta(\alpha, \varepsilon, k_{\max}(r), r)) = \max_{p_{\inf} \le p \le 1} pI(\beta(p,\alpha)). \tag{4.7.63}$$

by Theorem 4.7.2, Inequality (4.7.22), and Equation (4.7.60).

Furthermore, there exist $p(\alpha) \in (p_{\inf}, 1]$ and $\beta(p(\alpha), \alpha) := \beta(\alpha) \in (\alpha, \alpha_{\sup})$ such that

$$\beta(\alpha) = \frac{\alpha - (1 - p(\alpha))L}{p(\alpha)} \tag{4.7.64}$$

and

$$\max_{p_{\inf} \le p \le 1} pI(\beta(p,\alpha)) = p(\alpha) I(\beta(\alpha)) \tag{4.7.65}$$

because $p \to pI(\beta(p,\alpha))$ is continuous. We note that $p(\alpha) > 0$ because $p_{\inf} > 0$ by Lemma 4.7.3.

Therefore, there exist $p(\alpha) \in (p_{\inf}, 1]$ and $\beta(\alpha) \in (\alpha, \alpha_{\sup})$ (given by Equations (4.7.64) and (4.7.65)) such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \le \max_{p_{\inf} \le p \le 1} pI(\beta(p,\alpha)) = p(\alpha)I(\beta(\alpha)) = p(\alpha) \left(\sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, \mathrm{d}\eta + h(\eta) : \int f \, \mathrm{d}\eta \ge \beta(\alpha) \right\} \right)$$

by Inequality (4.7.63) and Equations (4.7.64) and (4.7.65). ■

By Equation (4.7.64), $\beta(\alpha)$ tends to α as $p(\alpha)$ tends to 1. We will now aim to form a lower bound $\underline{R}(\alpha)$ for our large deviation principle.

4.8 Preparation and the Proof of the Lower Bound

Fix an $\alpha \in (L, \alpha_{\sup})$. Now, we recall the set X_α^n. For each $n \in \mathbb{N}$, define the set

$$X_\alpha^n := \left\{ x = (x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots) \in \Sigma_A : \frac{\sum_{i=0}^{n-1} f(\sigma^i(x))}{n} \ge \alpha \right\}.$$

Our aim is to form a lower bound (see Theorem 4.8.6) for

$$\liminf_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n).$$

We will use Egoroff's Theorem, the escaping set Ω_λ, the transition matrix, and Proposition 4.8.1 (which was a covering argument) to construct two subsets of Σ_A. These subsets will be used to form the set $K^n_{\alpha,\widetilde{M}} \subset X_\alpha^n$. We will later choose $\widetilde{M}$ in Inequality (4.8.3). We outline the steps for constructing this set and forming a lower bound.

1. First, we define the variables for our various sets. We choose an $\widetilde{\varepsilon} > 0$ and then, we consider a value $\widetilde{M} \in \mathbb{N}$ (see Inequality (4.8.3)). Next, we consider the value $p(\alpha)$ (see Equations (4.7.64) and (4.7.65)). By using our chosen α and ϵ and the value $p(\alpha)$, we take the value $\widehat{\alpha} := \widehat{\alpha}(\widetilde{\varepsilon}) \in (\alpha, \alpha_{\sup})$ (see Proposition 4.8.1):

$$\widehat{\alpha}(\widetilde{\varepsilon}) := \frac{\alpha - (1 - p(\alpha))(L - \widetilde{\varepsilon})}{p(\alpha)}. \tag{4.8.1}$$

   Then, we use these values to form a set $B^{\lfloor np(\alpha)\rfloor, n}_{\widehat{\alpha}, \widetilde{M}}$ (explained in the next step). Equation (4.8.1) motivates the construction of the following key sets for $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$.

2. Consider

$$X^{\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}} = \left\{ x \in \Sigma_A : \frac{1}{\lfloor np(\alpha)\rfloor} S_{\lfloor np(\alpha)\rfloor} f(x) \geq \widehat{\alpha}. \right\}$$

   In Subsection 4.8.3, we construct a subset $B^{\lfloor np(\alpha)\rfloor, n}_{\widehat{\alpha}, \widetilde{M}} \subset X^{\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}}$.

3. Consider the set

$$X^{n - \lfloor np(\alpha)\rfloor}_{L - \widetilde{\varepsilon}} = \left\{ x \in \Sigma_A : \frac{1}{n - \lfloor np(\alpha)\rfloor} \sum_{i=0}^{n - \lfloor np(\alpha)\rfloor - 1} f(\sigma^i(x)) \geq L - \widetilde{\varepsilon} \right\}.$$

   In Subsection 4.8.4, we construct $A^{n - \lfloor np(\alpha)\rfloor, n}_{L - \widetilde{\varepsilon}, \widetilde{M}}$.

4. Because $\alpha = p(\alpha)\widehat{\alpha} + (1 - p)(L - \varepsilon)$, we form the subset

$$K^n_{\alpha,\widetilde{M}} := B^{\lfloor np(\alpha)\rfloor, n}_{\widehat{\alpha}, \widetilde{M}} \cap \sigma^{-\lfloor np(\alpha)\rfloor}(A^{n - \lfloor np(\alpha)\rfloor, n}_{L - \widetilde{\varepsilon}, \widetilde{M}}) \subset X^n_\alpha.$$

5. Finally, we use measure and cardinality estimates for cylinders in $B^{\lfloor np(\alpha)\rfloor, n}_{\widehat{\alpha}, \widetilde{M}}$ and $A^{n - \lfloor np(\alpha)\rfloor, n}_{L - \widetilde{\varepsilon}, \widetilde{M}}$ to get a conditional variational principle as our lower bound.

4.8.1 Constructing Subsets of X^n_α

Fix $\alpha > L$. We will now build a set $\widehat{X}^{n, \lfloor np(\alpha)\rfloor}_{\alpha, \widetilde{M}} \subset X^n_\alpha$. The construction of these subsets is motivated by the transient behaviour of $m-$typical sequences (proven in Theorem 4.1.4) and the value of their Birkhoff averages (proven in Proposition 4.1.5). First, we define necessary variables and values for this subset.

For each $N \in \mathbb{N}$, consider the sequence $\bar{N} := (N, N, N, \ldots) \in \Sigma_A$. Denote

$$L := \lim_{N \to \infty} f(\bar{N}).$$

Note that

$$\lim_{n \to \infty} \frac{S_n f(x)}{n} = L \tag{4.8.2}$$

for $m-$typical $x \in \Sigma_A$ by Theorem 4.1.4. This motivates our choice of $\widetilde{M} \in \mathbb{N}$ (see Inequality (4.8.3)). Let $\widetilde{\varepsilon} > 0$. Consider an arbitrary sequence $(J, x_2, ..., x_n, ...) \in [J]$ for each $J \in \mathbb{N}$. There exists a value $\widetilde{M} := \widetilde{M}(\widetilde{\varepsilon})$ such that for all $J \geq \widetilde{M}$,

$$f(\widehat{J}) \geq L - \widetilde{\varepsilon}. \tag{4.8.3}$$

Throughout Section 4.8, we will take the value $\widetilde{M} \in \mathbb{N}$ that satisfies Inequality (4.8.3).

Now, we will need to define the value $p(\alpha)$. Define the function I as

$$I(\gamma) := \sup_{\mu \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \mathrm{d}\mu + h(\mu) : \int f \mathrm{d}\mu \geq \gamma \right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$. Take the value

$$p_{\inf} = \frac{\alpha - L}{\alpha_{\sup} - L}.$$

Define the function β as

$$\beta(p, \alpha) := \frac{\alpha - (1-p)L}{p} \tag{4.8.4}$$

for each $p \in (p_{\inf}, 1]$. Consider the values $p(\alpha) \in (p_{\inf}, 1]$ and $\beta(p(\alpha), \alpha) := \beta(\alpha) \in (\alpha, \alpha_{\sup})$ such that

$$\beta(\alpha) = \frac{\alpha - (1-p(\alpha))L}{p(\alpha)} \tag{4.8.5}$$

and

$$\max_{p_{\inf} \leq p \leq 1} pI(\beta(p, \alpha)) = p(\alpha) I(\beta(\alpha)). \tag{4.8.6}$$

We proved the existence of these values in Theorem 4.7.6. We will only be using $p(\alpha)$, defined by Equations (4.7.47) and (4.7.48), in this section. We will see $\beta(\alpha)$ in our lower bound's proof (see Theorem 4.8.6). Now, we motivate our construction of $\widehat{X}^{n, \lfloor np(\alpha) \rfloor}_{\widetilde{M}}$.

The transient behaviour of $m-$typical sequences (see Theorem 4.1.4) motivates the construction of the following subset. Define the set

$$\widehat{X}^{n, \lfloor np(\alpha) \rfloor}_{\widetilde{M}} = \left\{ x \in \Sigma_A : x_{\lfloor np(\alpha) \rfloor} \leq \widetilde{M} \text{ and } x_i \geq \widetilde{M} + 1 \ \forall i \geq \lfloor np(\alpha) \rfloor + 1 \right\}. \tag{4.8.7}$$

and in turn, we take the subset

$$\widehat{X}^{n, \lfloor np(\alpha) \rfloor}_{\alpha, \widetilde{M}} = \widehat{X}^{n, \lfloor np(\alpha) \rfloor}_{\widetilde{M}} \cap X^n_\alpha. \tag{4.8.8}$$

We will use that

$$m(\widehat{X}^{n, \lfloor np(\alpha) \rfloor}_{\alpha, \widetilde{M}}) \leq m(X^n_\alpha)$$

to form the lower bound for our large deviation principle (see Theorem 4.8.6).

4.8.2 Covering $\widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}}$ and Outline of the Lower Bound's Argument

To form the lower bound for our large deviation principle (see Theorem 4.8.6), we need to construct a cover for $\widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}}$.

For each $\widehat{\alpha} \in (\alpha, \alpha_{\sup})$, let

$$\widetilde{X}^{\lfloor np(\alpha)\rfloor,\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}} := \widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\widetilde{M}} \cap X^{\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}}.$$

We will choose $\widehat{\alpha}$ according to the following proposition. The proposition shows the existence of a cover $\widetilde{X}^{\lfloor np(\alpha)\rfloor,\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}}$ for $\widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}}$. This cover will be key to the construction of $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$.

Proposition 4.8.1. *Let $\widetilde{\varepsilon} > 0$. There exists a value $\widehat{\alpha} := \widehat{\alpha}(\widetilde{\varepsilon}) \in (\alpha, \alpha_{\sup})$ such that*

1.

$$\widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}} \subset \widetilde{X}^{\lfloor np(\alpha)\rfloor,\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}} \textit{ and}$$

2.

$$p(\alpha)\widehat{\alpha}(\widetilde{\varepsilon}) + (1-p(\alpha))(L-\widetilde{\varepsilon}) = \alpha$$

or alternatively,

$$\widehat{\alpha}(\widetilde{\varepsilon}) = \frac{\alpha-(1-p(\alpha))(L-\widetilde{\varepsilon})}{p(\alpha)}$$

Proof. Take any $x = (x_1, \ldots, x_{\lfloor np(\alpha)\rfloor}, \ldots) \in \widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}}$. Then,

$$x_i \geq \widetilde{M} + 1 \tag{4.8.9}$$

for each $i \in \{\lfloor np(\alpha)\rfloor + 1, \ldots, n-1, n\}$. Let $\widetilde{\varepsilon} > 0$. Because of our choice of $\widetilde{M}$ (see Inequality (4.8.3)), $p(\alpha) \in (0,1]$, and Inequality (4.8.9),

$$\frac{1}{n-\lfloor np(\alpha)\rfloor} \sum_{i=\lfloor np(\alpha)\rfloor}^{n-1} f(\sigma^i(x)) \geq L - \widetilde{\varepsilon} \tag{4.8.10}$$

for all $x \in \widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}}$.

Because of Inequality (4.8.10) and $x \in \widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}} \subset X^n_\alpha$, there exists an $\widehat{\alpha} := \widehat{\alpha}(\widetilde{\varepsilon}) \in (\alpha, \alpha_{\sup})$ that satisfies

$$\frac{1}{\lfloor np(\alpha)\rfloor} \sum_{i=0}^{\lfloor np(\alpha)\rfloor-1} f(\sigma^i(x)) \geq \widehat{\alpha}(\widetilde{\varepsilon}). \tag{4.8.11}$$

Furthermore,

$$\sum_{i=0}^{n-1} f(\sigma^i(x)) \geq \lfloor np(\alpha)\rfloor\, \widehat{\alpha}(\widetilde{\varepsilon}) + (n-\lfloor np(\alpha)\rfloor)(L-\widetilde{\varepsilon}), \tag{4.8.12}$$

for all $x \in \widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}}$ because of Inequalities (4.8.10) and (4.8.11). Furthermore, we choose $\widehat{\alpha} := \widehat{\alpha}(\widetilde{\varepsilon}) \in (\alpha, \alpha_{\sup})$ as follows:

$$p(\alpha)\widehat{\alpha}(\widetilde{\varepsilon}) + (1-p(\alpha))(L-\widetilde{\varepsilon}) = \alpha \tag{4.8.13}$$

or alternatively,

$$\widehat{\alpha}(\widetilde{\varepsilon}) = \frac{\alpha-(1-p(\alpha))(L-\widetilde{\varepsilon})}{p(\alpha)}$$

by Inequality (4.8.13). Then,

$$\sum_{i=0}^{n-1} f(\sigma^i(x)) \geq n\alpha$$

by Inequality (4.8.12) and Equation (4.8.13). Therefore,

$$\widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}} \subset \widetilde{X}^{\lfloor np(\alpha)\rfloor,\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}}$$

by Inequalities (4.8.11) and (4.8.12) and Equation (4.8.13). ■

We aim to form a lower bound $\underline{R}(\alpha)$ (see Theorem 4.8.6) for

$$\liminf_{n\to\infty} \frac{1}{n} \log m(X^n_\alpha).$$

To form this bound, we will bound $m(X^n_\alpha)$ below. To do this, we will now build the key sets, $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$ and $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$. These sets will be used to form $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$. First, we construct the set $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}} \subset \widetilde{X}^{\lfloor np(\alpha)\rfloor,\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}}$.

4.8.3 Constructing $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$ and Measure and Cardinality Estimates

Now, we will construct one of the two sets that will form $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$. Take our fixed $\alpha \in (L, \alpha_{\sup})$. Consider the value $\widehat{\alpha} \in (\alpha, \alpha_{\sup})$ from Proposition 4.8.1. We outline the results of this subsection.

1. By using Egoroff's Theorem, the Birkhoff Ergodic Theorem, the Shannon-McMillan-Breiman Theorem, and Proposition 4.8.1, we construct the set $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}} \subset \widetilde{X}^{\lfloor np(\alpha)\rfloor,\lfloor np(\alpha)\rfloor}_{\widehat{\alpha}}$. Each element of this set has a Birkhoff average greater than $\widehat{\alpha}$. We use an invariant measure $\eta \in M_\sigma(\Sigma_A)$ in this construction.

2. Consider the set $B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ of $\lfloor np(\alpha)\rfloor-$cylinders that contain $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$. We bound the $\eta-$measure of each cylinder above and the $m-$measure of each cylinder below. Then, we form a lower bound for the cardinality of this set.

The construction of $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$ will help us form $K^n_{\alpha,\widetilde{M}}$ and the results for $B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ will help us bound $m(X^n_\alpha)$ below. We move on to forming the set $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$.

First, we introduce an invariant measure. Let $\eta \in M_\sigma(\Sigma_A)$ be ergodic such that

$$\eta([i]) > 0 \text{ for each } i \leq \widetilde{M} \text{ and } \int f \, d\eta \geq \widehat{\alpha}. \tag{4.8.14}$$

For each sufficiently small $\bar{\xi} > 0$ and

$$\delta := \frac{\widehat{\alpha} - \int f \, d\eta}{2} > 0, \tag{4.8.15}$$

there exists a set $B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n} \subset \widetilde{X}_{\widehat{\alpha}}^{\lfloor np(\alpha)\rfloor,\lfloor np(\alpha)\rfloor}$, with measure

$$1 \geq \eta\left(B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n}\right) \geq \frac{9}{10},$$

such that each $x = (x_1, ..., x_{\lfloor np(\alpha)\rfloor}, ...) \in B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n}$ satisfies

$$\frac{1}{\lfloor np(\alpha)\rfloor} \sum_{i=0}^{\lfloor np(\alpha)\rfloor - 1} f(\sigma^i(x)) \geq \widehat{\alpha} + \delta \tag{4.8.16}$$

$$\int \phi_\lambda \, d\eta - \bar{\xi} < \frac{1}{\lfloor np(\alpha)\rfloor} \sum_{i=0}^{\lfloor np(\alpha)\rfloor - 1} \phi_\lambda(\sigma^i(x)) < \int \phi_\lambda \, d\eta + \bar{\xi}, \text{ and} \tag{4.8.17}$$

$$h(\eta) - \bar{\xi} \leq -\frac{1}{\lfloor np(\alpha)\rfloor} \log \eta([x_1, ..., x_{\lfloor np(\alpha)\rfloor}]) \leq h(\eta) + \bar{\xi} \tag{4.8.18}$$

by Egoroff's Theorem, the Birkhoff Ergodic Theorem, and the Shannon-McMillan-Breiman Theorem.

We need results for cylinders that contain elements of $B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n}$. Hence, we define the set of cylinders

$$B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,c} := \left\{[x_1, ..., x_{\lfloor np(\alpha)\rfloor}] \subset \Sigma_A : [x_1, ..., x_{\lfloor np(\alpha)\rfloor}] \cap B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n} \neq \emptyset\right\}. \tag{4.8.19}$$

Now, we will begin taking the necessary steps to bound the cardinality of $B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,c}$ below. First, we bound $\eta([x_1, ..., x_{\lfloor np(\alpha)\rfloor}])$ above for each $[x_1, ..., x_{\lfloor np(\alpha)\rfloor}] \in B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,c}$.

Lemma 4.8.2. *Fix a sufficiently small $\bar{\xi} > 0$. For each cylinder* $[x_1, ..., x_{\lfloor np(\alpha)\rfloor}] \in B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,c}$,

$$\eta([x_1, ..., x_{\lfloor np(\alpha)\rfloor}]) \leq \exp(-\lfloor np(\alpha)\rfloor(h(\eta) - \bar{\xi})). \tag{4.8.20}$$

Proof. Take a sufficiently small $\bar{\xi} > 0$. For each $[x_1, ..., x_{\lfloor np(\alpha)\rfloor}] \in B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,c}$, Inequality (4.8.18) states that

$$h(\eta) - \bar{\xi} \leq -\frac{1}{\lfloor np(\alpha)\rfloor} \log \eta([x_1, ..., x_{\lfloor np(\alpha)\rfloor}])$$

Therefore,

$$\eta([x_1, ..., x_{\lfloor np(\alpha)\rfloor}]) \leq \exp(-\lfloor np(\alpha)\rfloor(h(\eta) - \bar{\xi})).$$

■

Lemma 4.8.2 will be used to bound the cardinality of $B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ below. We will need this estimate because $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$ will be used to construct a set $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$. Denote $|S|$ as the cardinality of a set $S \subset \Sigma_A$.

Proposition 4.8.3. *We find that*

$$|B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}| \geq \frac{9}{10}\exp(\lfloor np(\alpha)\rfloor(h(\eta)-\bar{\xi}))$$

for a sufficiently small $\bar{\xi} > 0$.

Proof. By assumption,

(4.8.21)

$$|B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}| \max_{[x_1,...,x_{\lfloor np(\alpha)\rfloor}]\subset B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}} \eta([x_1,...,x_{\lfloor np(\alpha)\rfloor}]) \geq \sum_{[x_1,...,x_{\lfloor np(\alpha)\rfloor}]\subset B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}} \eta([x_1,...,x_{\lfloor np(\alpha)\rfloor}]) = \eta(B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}) \geq \frac{9}{10}.$$

Then,

$$|B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}| \geq \left(\frac{9}{10}\right)\left(\max_{[x_1,...,x_{\lfloor np(\alpha)\rfloor}]\subset B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}} \eta([x_1,...,x_{\lfloor np(\alpha)\rfloor}])\right)^{-1}. \tag{4.8.22}$$

Now, we use our bound for $\eta([x_1,...,x_{\lfloor np(\alpha)\rfloor}])$. Take a sufficiently small $\bar{\xi} > 0$. By Lemma 4.8.2,

$$\left(\max_{[x_1,...,x_{\lfloor np(\alpha)\rfloor}]\subset B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}} \eta([x_1,...,x_{\lfloor np(\alpha)\rfloor}])\right)^{-1} \geq \exp(\lfloor np(\alpha)\rfloor(h(\eta)-\bar{\xi})) \tag{4.8.23}$$

Therefore,

$$|B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}| \geq \frac{9}{10}\exp(\lfloor np(\alpha)\rfloor(h(\eta)-\bar{\xi})) \tag{4.8.24}$$

by Inequalities (4.8.22) and (4.8.23). ■

We also need to bound $m([x_1,...,x_{\lfloor np(\alpha)\rfloor}])$ below for each $[x_1,...,x_{\lfloor np(\alpha)\rfloor}] \in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ because it will be useful for our large deviation argument.

Proposition 4.8.4. *Take a sufficiently small $\bar{\xi} > 0$. For each cylinder $[x_1,...,x_{\lfloor np(\alpha)\rfloor}] \in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$,*

$$\lambda^{\widetilde{M}-1}\exp(\lfloor np(\alpha)\rfloor(-\lambda(\eta)-\bar{\xi})) \leq m([x_1,...,x_{\lfloor np(\alpha)\rfloor}]). \tag{4.8.25}$$

Proof. Take any $x = (x_1,...,x_{\lfloor np(\alpha)\rfloor},...) \subset B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$. Then, consider the cylinder $[x_1,...,x_{\lfloor np(\alpha)\rfloor}] \subset \Sigma_A$. Its $\lfloor np(\alpha)\rfloor$−th symbol

$$x_{\lfloor np(\alpha)\rfloor} \leq \widetilde{M}. \tag{4.8.26}$$

Hence, by construction of the function N (see Equation (4.2.1)),

$$N([x_1,...,x_{\lfloor np(\alpha)\rfloor-1},x_{\lfloor np(\alpha)\rfloor}]) \geq N([x_1,...,x_{\lfloor np(\alpha)\rfloor-1}]). \tag{4.8.27}$$

We find the bound

$$m([x_1,...,x_{\lfloor np(\alpha)\rfloor}]) = \lambda^{N([x_1,...,x_{\lfloor np(\alpha)\rfloor-1}])+x_{\lfloor np(\alpha)\rfloor}-1}(1-\lambda)^{\lfloor np(\alpha)\rfloor}.$$

$$\geq \lambda^{x_{\lfloor np(\alpha)\rfloor}-1}\lambda^{N([x_1,...,x_{\lfloor np(\alpha)\rfloor-1},x_{\lfloor np(\alpha)\rfloor}])}(1-\lambda)^{\lfloor np(\alpha)\rfloor}$$

$$\geq \lambda^{\widetilde{M}-1}\lambda^{N([x_1,...,x_{\lfloor np(\alpha)\rfloor-1},x_{\lfloor np(\alpha)\rfloor}])}(1-\lambda)^{\lfloor np(\alpha)\rfloor} = \lambda^{\widetilde{M}-1}\exp(S_{\lfloor np(\alpha)\rfloor}\phi_\lambda(x)) \tag{4.8.28}$$

by Inequalities (4.8.27) and (4.8.26) and Equations (4.2.2) and (4.2.3).

Take a sufficiently small $\bar{\xi} > 0$. By Inequality (4.8.17),

$$\exp\left(\lfloor np(\alpha)\rfloor\left[\int \phi_\lambda\, d\eta - \bar{\xi}\right]\right) < \exp(S_{\lfloor np(\alpha)\rfloor}\phi_\lambda(x)). \tag{4.8.29}$$

Therefore,

$$\lambda^{\widetilde{M}-1}\exp\left(\lfloor np(\alpha)\rfloor\left[\int \phi_\lambda\, d\eta - \bar{\xi}\right]\right) \leq m([x_1,...,x_{\lfloor np(\alpha)\rfloor}])$$

for each $[x_1,...,x_{\lfloor np(\alpha)\rfloor}] \in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ by applying Inequality (4.8.29) to Inequality (4.8.28). ■

Consider $\bar{N} := (N,N,N,...) \in \Sigma_A$ for each $N \in \mathbb{N}$. Recall that

$$L := \lim_{N\to\infty} f(\bar{N}).$$

To build our set $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$, we will construct a set $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}} := \sigma^{\lfloor np(\alpha)\rfloor}\left(B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}\right) \cap X^{n-\lfloor np(\alpha)\rfloor}_{L-\widetilde{\varepsilon}}$ because of the $m-$typical behaviour of sequences in Σ_A (as stated in Theorem 4.1.4) and the Birkhoff averages of these sequences (see Proposition 4.1.5), and our expression for α (see Proposition 4.8.1).

The set $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$ is key because we will form (see Equation 4.8.36):

$$K^n_{\alpha,\widetilde{M}} := B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}} \cap \sigma^{-\lfloor np(\alpha)\rfloor}\left(A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}\right)$$

and later show that $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$ (see Equation (4.8.37)).

In the next subsection, we will build $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$.

4.8.4 Constructing $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$ and Bounding Its Measure Below

We will construct $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$ and find a lower bound for its measure. To construct this subset, we will use that $m-$typical sequences are transient and the Birkhoff averages of these sequences (respectively, see Theorem 4.1.4 and Proposition 4.1.5), our transition matrix (see Equation (4.1.2)), and Egoroff's Theorem. Now, we revisit various results to help us build $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$.

Proposition 4.1.5 states that

$$\lim_{n\to\infty}\frac{1}{n}S_n f(x) = L$$

for each $m-$typical $x \in \Sigma_A$. Consider the set

$$X_{L-\widetilde{\varepsilon}}^{n-\lfloor np(\alpha)\rfloor} = \left\{ x \in \Sigma_A : \frac{1}{n-\lfloor np(\alpha)\rfloor} \sum_{i=0}^{n-\lfloor np(\alpha)\rfloor-1} f(\sigma^i(x)) \geq L - \widetilde{\varepsilon} \right\}. \tag{4.8.30}$$

Because the Birkhoff average of each $m-$typical sequence equals L (see Proposition 4.1.5), there exists a sufficiently small $\delta(n) > 0$ such that

$$m(X_{L-\widetilde{\varepsilon}}^{n-\lfloor np(\alpha)\rfloor}) \geq 1 - \delta(n). \tag{4.8.31}$$

Hence, we will use $X_{L-\widetilde{\varepsilon}}^{n-\lfloor np(\alpha)\rfloor}$ to construct $A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,n}$.

To build $A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,n}$, we find the result:

$$B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n} \cap \sigma^{-\lfloor np(\alpha)\rfloor}(X_{L-\widetilde{\varepsilon}}^{n-\lfloor np(\alpha)\rfloor}) \neq \emptyset \tag{4.8.32}$$

because of our choice of $\widetilde{M}$ (see Inequality (4.8.3)), the definition of $X_{L-\widetilde{\varepsilon}}^{n-\lfloor np(\alpha)\rfloor}$, and our transition matrix (see Equation (4.1.2)).

Because of Egoroff's Theorem, $X_{L-\widetilde{\varepsilon}}^{n-\lfloor np(\alpha)\rfloor}$ is nearly a full measure set (see Inequality (4.8.31)), and $B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n}$ and $\sigma^{-\lfloor np(\alpha)\rfloor}(X_{L-\widetilde{\varepsilon}}^{n-\lfloor np(\alpha)\rfloor})$ have non-empty intersection, Equation (4.8.32), there exists a set

$$A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,n} = \sigma^{\lfloor np(\alpha)\rfloor}\left(B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n}\right) \cap \bar{X}_{L-\widetilde{\varepsilon}}^{n-\lfloor np(\alpha)\rfloor} \tag{4.8.33}$$

with measure

$$m(A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,n}) \geq \frac{1}{2}. \tag{4.8.34}$$

We will need to consider a set of cylinders that contain elements of $A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,n}$. Define the set of cylinders

$$A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,c} := \left\{ [x_{\lfloor np(\alpha)\rfloor+1},\ldots,x_n] \subset \Sigma_A : [x_{\lfloor np(\alpha)\rfloor+1},\ldots,x_n] \cap A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,n} \neq \emptyset \right\}.$$

We started the indexing of these cylinders at $\lfloor np(\alpha)\rfloor + 1$ because it will make the construction of $K_{\alpha,\widetilde{M}}^n$ simpler later. We find that

$$m\left(A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,c}\right) \geq m\left(A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,n}\right) \geq \frac{1}{2} \tag{4.8.35}$$

by Inequality (4.8.34).

Because we have constructed $B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,n}$ and $A_{L-\widetilde{\varepsilon},\widetilde{M}}^{n-\lfloor np(\alpha)\rfloor,n}$ and analysed their properties, we are ready to construct a key set $K_{\alpha,\widetilde{M}}^n \subset X_\alpha^n$.

4.8.5 Construction of $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$

Consider our fixed $\alpha \in (L, \alpha_{\sup})$. We will build a set $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$ by concatenating elements of $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$ and $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$. Then, we consider the set of $n-$cylinders $K^{n,c}_{\alpha,\widetilde{M}}$ that contain $K^n_{\alpha,\widetilde{M}}$.

Now, we revisit a few results before building $K^n_{\alpha,\widetilde{M}}$. Let $\widetilde{\varepsilon} > 0$. Recall the sets

$$X^n_\alpha := \left\{ x = (x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots) \in \Sigma_A : \frac{\sum_{i=0}^{n-1} f(\sigma^i(x))}{n} \geq \alpha \right\},$$

$$\widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\widetilde{M}} = \left\{ x \in \Sigma_A : x_{\lfloor np(\alpha)\rfloor} \leq \widetilde{M} \text{ and } x_i \geq \widetilde{M}+1 \ \forall i \in \{\lfloor np(\alpha)\rfloor+1, \dots, n-1, n\} \right\},$$

$$\widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}} = \widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\widetilde{M}} \cap X^n_\alpha,$$

$$\text{and } X^{n-\lfloor np(\alpha)\rfloor}_{L-\widetilde{\varepsilon}} = \left\{ x \in \Sigma_A : \frac{1}{n-\lfloor np(\alpha)\rfloor} \sum_{i=0}^{n-\lfloor np(\alpha)\rfloor-1} f(\sigma^i(x)) \geq L - \widetilde{\varepsilon} \right\}.$$

There exists a $\widehat{\alpha} := \widehat{\alpha}(\widetilde{\varepsilon}) \in (\alpha, \alpha_{\sup})$ such that

$$\widehat{\alpha}(\widetilde{\varepsilon}) = \frac{\alpha - (1-p(\alpha))(L-\widetilde{\varepsilon})}{p(\alpha)}$$

by Proposition 4.8.1. Hence, we use Proposition 4.8.1 to construct $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$ from the sets $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}} \subset \widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}}$ (from Subsection 4.8.3) such that

$$\frac{S_{\lfloor np(\alpha)\rfloor} f(x)}{\lfloor np(\alpha)\rfloor} > \widehat{\alpha}$$

for each $x \in B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$ and

$$\frac{S_{n-\lfloor np(\alpha)\rfloor} f(y)}{n-\lfloor np(\alpha)\rfloor} \geq L - \widetilde{\varepsilon}$$

for each $y \in A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$ (from Subsection 4.8.4).

Let

$$K^n_{\alpha,\widetilde{M}} := B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}} \cap \sigma^{-\lfloor np(\alpha)\rfloor}\left(A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}} \right). \tag{4.8.36}$$

We can construct $K^n_{\alpha,\widetilde{M}}$ because of our transition matrix (see Equation (4.1.2)) and Birkhoff averages of $m-$typical points equal L (see Proposition 4.1.5). Furthermore,

$$K^n_{\alpha,\widetilde{M}} \subset \widehat{X}^{n,\lfloor np(\alpha)\rfloor}_{\alpha,\widetilde{M}} \subset X^n_\alpha \tag{4.8.37}$$

by definition of $K^n_{\alpha,\widetilde{M}}$ (see Equation (4.8.36)). Recall that

$$B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}} := \left\{ [x_1, \dots, x_{\lfloor np(\alpha)\rfloor}] \subset \Sigma_A : [x_1, \dots, x_{\lfloor np(\alpha)\rfloor}] \cap B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}} \neq \emptyset \right\}$$

$$\text{and } A^{n-\lfloor np(\alpha)\rfloor,c}_{L-\widetilde{\varepsilon},\widetilde{M}} := \left\{ [x_{\lfloor np(\alpha)\rfloor+1}, \dots, x_n] \subset \Sigma_A : [x_{\lfloor np(\alpha)\rfloor+1}, \dots, x_n] \cap A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}} \neq \emptyset \right\}.$$

We will need to consider the set of $n-$cylinders that contain elements of $K^n_{\alpha,\widetilde{M}}$. Define the set

$$K^{n,c}_{\alpha,\widetilde{M}} := \left\{[x_1,...,x_n] \subset \Sigma_A : [x_1,...,x_n] \cap K^n_{\alpha,\widetilde{M}} \neq \emptyset\right\}.$$

Proposition 4.8.5 states that the set of $n-$cylinders in $K^{n,c}_{\alpha,\widetilde{M}}$ is made of all allowable concatenations of $\lfloor np(\alpha)\rfloor-$cylinders in $B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ and $(n-\lfloor np(\alpha)\rfloor)-$cylinders in $A^{n-\lfloor np(\alpha)\rfloor,c}_{L-\widetilde{\varepsilon},\widetilde{M}}$.

Proposition 4.8.5. *For each $n-$cylinder $[x_1,...,x_n] \in K^{n,c}_{\alpha,\widetilde{M}}$, there exists a unique pair of cylinders $[x_1,...,x_{\lfloor np(\alpha)\rfloor}] \in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ and $[x_{\lfloor np(\alpha)\rfloor+1},...x_n] \in A^{n-\lfloor np(\alpha)\rfloor,c}_{L-\widetilde{\varepsilon},\widetilde{M}}$ such that*

$$[x_1,...,x_n] = [x_1,...,x_{\lfloor np(\alpha)\rfloor}] \cap \sigma^{-\lfloor np(\alpha)\rfloor}([x_{\lfloor np(\alpha)\rfloor+1},...x_n]).$$

Furthermore, any pair of cylinders $[y_1,...,y_{\lfloor np(\alpha)\rfloor}] \in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ and $[y_{\lfloor np(\alpha)\rfloor+1},...y_n] \in A^{n-\lfloor np(\alpha)\rfloor,c}_{L-\widetilde{\varepsilon},\widetilde{M}}$ can be concatenated to form a cylinder

$$[y_1,...,y_n] = [y_1,...,y_{\lfloor np(\alpha)\rfloor}] \cap \sigma^{-\lfloor np(\alpha)\rfloor}([y_{\lfloor np(\alpha)\rfloor+1},...y_n]) \in K^{n,c}_{\alpha,\widetilde{M}}.$$

Proof. Because the $\lfloor np(\alpha)\rfloor-$th symbol

$$x_{\lfloor np(\alpha)\rfloor} \leq \widetilde{M} \tag{4.8.38}$$

for each $x \in [x_1,....,x_{\lfloor np(\alpha)\rfloor}] \in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ and the 1st symbol

$$\widetilde{M}+1 \leq x_{\lfloor np(\alpha)\rfloor+1} < \infty, \tag{4.8.39}$$

for each $x \in [x_{\lfloor np(\alpha)\rfloor+1},...,x_n] \in A^{n-\lfloor np(\alpha)\rfloor,c}_{L-\widetilde{\varepsilon},\widetilde{M}}$, the transition matrix (see Equation (4.1.2)) has the entry

$$a_{x_{\lfloor np(\alpha)\rfloor},x_{\lfloor np(\alpha)\rfloor+1}} = 1. \tag{4.8.40}$$

Therefore, there exists a unique pair of cylinders $[x_1,...,x_{\lfloor np(\alpha)\rfloor}] \in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}$ and $[x_{\lfloor np(\alpha)\rfloor+1},...x_n] \in A^{n-\lfloor np(\alpha)\rfloor,c}_{L-\widetilde{\varepsilon},\widetilde{M}}$ such that

$$[x_1,...,x_n] = [x_1,...,x_{\lfloor np(\alpha)\rfloor}] \cap \sigma^{-\lfloor np(\alpha)\rfloor}([x_{\lfloor np(\alpha)\rfloor+1},...x_n])$$

for each $[x_1,...,x_n] \in K^{n,c}_{\alpha,\widetilde{M}}$ because of Equation (4.8.40) and

$$K^n_{\alpha,\widetilde{M}} = B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}} \cap \sigma^{-\lfloor np(\alpha)\rfloor}(A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}). \tag{4.8.41}$$

Furthermore, we find the latter result by Inequalities (4.8.38) and (4.8.39) and Equations (4.8.40) and (4.8.41). ■

4.8.6 Statement and Proof of the Lower Bound

We will now form a lower bound

$$\liminf_{n\to\infty}\frac{1}{n}\log m(X_\alpha^n)\geq \underline{R}(\alpha).$$

We will soon find that this bound uses the following function. Define the function I by

$$I(\gamma):=\sup_{\mu\in M_\sigma(\Sigma_A)}\left\{\int \phi_\lambda \mathrm{d}\mu + h(\mu) : \int f\,\mathrm{d}\mu \geq \gamma\right\}$$

for each $\gamma\in(\alpha,\alpha_{\sup})$. We will also need the values $p(\alpha)$ and $\beta(\alpha)$ for our lower bound. First, we must define the following. Recall the value

$$p_{\inf}:=\frac{\alpha_{\sup}-L}{\alpha-L}.$$

Define the function β as

$$\beta(p,\alpha)=\frac{\alpha-(1-p)L}{p} \tag{4.8.42}$$

for each $p\in(p_{\inf},1]$. Consider the values $p(\alpha)\in(p_{\inf},1]$ and $\beta(p(\alpha),\alpha):=\beta(\alpha)\in(\alpha,\alpha_{\sup})$ such that

$$\beta(p(\alpha),\alpha)=\frac{\alpha-(1-p(\alpha))L}{p(\alpha)} \tag{4.8.43}$$

and

$$\max_{p_{\inf}\leq p\leq 1} pI(\beta(p,\alpha))=p(\alpha)I(\beta(\alpha)). \tag{4.8.44}$$

We proved the existence of these values in Theorem 4.7.6. Finally, we state the lower bound for our large deviation principle.

Theorem 4.8.6. *Fix $\lambda\in\left(\frac{1}{2},1\right)$. Recall the map T_λ given by Equation* (4.1.1)*, the shift space* (Σ_A,σ)*, and the coding map $\pi:\Sigma_A\to(0,1]$. Let $\phi_\lambda:=-\log|T_\lambda^{'}\circ\pi|$. Take $\bar{N}:=(N,N,N,\ldots)\in\Sigma_A$ for each $N\in\mathbb{N}$. Assume that $f:\Sigma_A\to\mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty}f(\bar{N})\in(-\infty,\infty)$. Recall that*

$$L:=\lim_{N\to\infty}f(\bar{N}) \text{ and } \alpha_{\sup}:=\sup_{\nu\in M_\sigma(\Sigma_A)}\left\{\int f\,\mathrm{d}\nu\right\}.$$

Fix $\alpha\in\left(L,\alpha_{\sup}\right)$. Recall the value

$$p_{\inf}:=\frac{\alpha_{\sup}-L}{\alpha-L}.$$

Then, there exist a $p(\alpha)\in(p_{\inf},1]$ and $\beta(\alpha)\in(\alpha,\alpha_{\sup})$ (see Equations (4.8.42), (4.8.43)*, and* (4.8.44)*) such that*

$$\liminf_{n\to\infty}\frac{1}{n}\log m(X_\alpha^n)\geq p(\alpha)\sup_{\eta\in M_\sigma(\Sigma_A)}\left\{\int \phi_\lambda\,\mathrm{d}\eta + h(\eta) : \int f\,\mathrm{d}\eta\geq\beta(\alpha)\right\}.$$

Proof. To find the lower bound for

$$\liminf_{n\to\infty}\frac{1}{n}\log m(X_\alpha^n),$$

we will bound $m(X_\alpha^n)$ below.

We outline the steps of our proof.

1. Respectively, see Inequalities (4.8.16), (4.8.17), and (4.8.18) for the construction of $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$ and Equations (4.8.30) and (4.8.33) for the construction of $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$. Because

$$K^n_{\alpha,\widetilde{M}} = B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}} \cap \sigma^{-\lfloor np(\alpha)\rfloor}(A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}),$$

We will use that $K^n_{\alpha,\widetilde{M}} \subset X^n_\alpha$ to bound $m(X^n_\alpha)$ below. To find this lower bound, we will use results for cylinders in $B^{\lfloor np(\alpha)\rfloor,n}_{\widehat{\alpha},\widetilde{M}}$ and $A^{n-\lfloor np(\alpha)\rfloor,n}_{L-\widetilde{\varepsilon},\widetilde{M}}$.

2. Again, consider the value $\widehat{\alpha} := \widehat{\alpha}(\widetilde{\varepsilon}) \in (\alpha, \alpha_{\sup})$. See Proposition 4.8.1 for the proof of its existence. We prove that

$$\lim_{\widetilde{\varepsilon}\to 0}\widehat{\alpha}(\widetilde{\varepsilon}) = \beta(\alpha)$$

See Equations (4.8.43) and (4.8.44) for the definition of $\beta(\alpha)$. Finally, we let $\widetilde{\varepsilon}$ tend to 0 in Inequality (4.8.46). This gives us a weighted conditional variational principle as the lower bound for

$$\liminf_{n\to\infty}\frac{1}{n}\log m(X^n_\alpha).$$

Now, we proceed by bounding $m(X^n_\alpha)$ below. We find that

$$m(X^n_\alpha) \geq m(K^n_{\alpha,\widetilde{M}}) = m\left(\bigcup_{K^{n,c}_{\alpha,\widetilde{M}}}[x_1,\ldots,x_{\lfloor np(\alpha)\rfloor},x_{\lfloor np(\alpha)\rfloor+1},\ldots x_n]\right). \tag{4.8.45}$$

$$= \sum_{[x_1,\ldots,x_{\lfloor np(\alpha)\rfloor},x_{\lfloor np(\alpha)\rfloor+1},\ldots x_n]\in K^{n,c}_{\alpha,\widetilde{M}}} m([x_1,\ldots,x_{\lfloor np(\alpha)\rfloor},x_{\lfloor np(\alpha)\rfloor+1},\ldots x_n])$$

$$\geq \left(\sum_{[x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}]\in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}} m([x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}])\right)\left(\sum_{[x_{\lfloor np(\alpha)\rfloor+1},\ldots x_n]\in A^{n-\lfloor np(\alpha)\rfloor,c}_{L-\widetilde{\varepsilon},\widetilde{M}}} m([x_{\lfloor np(\alpha)\rfloor+1},\ldots x_n])\right)$$

$$= \left(\sum_{[x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}]\in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}} m([x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}])\right) m\left(A^{n-\lfloor np(\alpha)\rfloor,c}_{L-\widetilde{\varepsilon},\widetilde{M}}\right)$$

$$\geq \frac{1}{2}\left(\sum_{[x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}]\in B^{\lfloor np(\alpha)\rfloor,c}_{\widehat{\alpha},\widetilde{M}}} m([x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}])\right)$$

by Proposition (4.8.5), Lemma (4.2.3), Inequality (4.8.35), and Equation (4.8.37). Then,

$$m(X_\alpha^n) \geq \frac{1}{2}\left(\sum_{[x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}]\in B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,c}} m([x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}])\right)$$

$$\geq \frac{1}{2}|B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,c}| \inf_{[x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}]\in B_{\widehat{\alpha},\widetilde{M}}^{\lfloor np(\alpha)\rfloor,c}} m([x_1,\ldots,x_{\lfloor np(\alpha)\rfloor}])$$

$$\geq \left(\frac{1}{2}\right)\left(\frac{9}{10}\right)\exp(\lfloor np(\alpha)\rfloor(h(\eta)-\bar{\xi}))\lambda^{\widetilde{M}-1}\exp\left(\lfloor np(\alpha)\rfloor\left(\int \phi_\lambda \,\mathrm{d}\eta - \bar{\xi}\right)\right)$$

$$= \frac{9\lambda^{\widetilde{M}-1}}{20}\exp\left(\lfloor np(\alpha)\rfloor\left(\int \phi_\lambda \,\mathrm{d}\eta + h(\eta) - 2\bar{\xi}\right)\right) \tag{4.8.46}$$

by applying Propositions 4.8.4 and 4.8.3 to Inequality (4.8.45).

We will now form the weighted conditional variational principle. Let $\widetilde{\varepsilon} > 0$. For our chosen $p(\alpha) \in (p_{\inf}, 1]$ (given by Equations (4.8.43) and (4.8.44)), there exists a value $\widehat{\alpha} := \widehat{\alpha}(\widetilde{\varepsilon}) \in (\alpha, \alpha_{\sup})$ such that

$$p(\alpha)\widehat{\alpha}(\widetilde{\varepsilon}) + (1-p(\alpha))(L-\widetilde{\varepsilon}) = \alpha \tag{4.8.47}$$

or alternatively,

$$\widehat{\alpha}(\widetilde{\varepsilon}) = \frac{\alpha - (1-p(\alpha))(L-\widetilde{\varepsilon})}{p(\alpha)}. \tag{4.8.48}$$

We showed the existence of $\widehat{\alpha}(\widetilde{\varepsilon})$ in Proposition 4.8.1.

We will prove that the following value, $\beta(\alpha)$, is the limit for the sequence of $\widehat{\alpha}(\widetilde{\varepsilon})$. For our $p(\alpha) \in (p_{\inf}, 1]$ (see Equations (4.8.42), (4.8.43), and (4.8.44)), there exists a value $\beta(\alpha)$ such that

$$p(\alpha)\beta(\alpha) + (1-p(\alpha))L = \alpha \tag{4.8.49}$$

or alternatively,

$$\beta(\alpha) = \frac{\alpha - (1-p(\alpha))L}{p(\alpha)} \tag{4.8.50}$$

Now, we will find the limit for the sequence of $\widehat{\alpha}(\widetilde{\varepsilon})$. By Equations (4.8.48) and (4.8.50),

$$\beta(\alpha) = \frac{\alpha - (1-p(\alpha))L}{p(\alpha)} = \lim_{\widetilde{\varepsilon}\to 0}\frac{\alpha - (1-p(\alpha))(L-\widetilde{\varepsilon})}{p(\alpha)} = \lim_{\widetilde{\varepsilon}\to 0}\widehat{\alpha}(\widetilde{\varepsilon}). \tag{4.8.51}$$

Note that $\widetilde{M} := \widetilde{M}(\widetilde{\varepsilon})$. Then,

$$\lim_{\widetilde{\varepsilon}\to 0}\widetilde{M}(\widetilde{\varepsilon}) = \infty \tag{4.8.52}$$

by Inequality (4.8.3) and neither $\widetilde{\varepsilon}$ nor $\widetilde{M}$ depend on n.

Therefore, there exist $p(\alpha) \in (p_{\inf}, 1]$ (see Equations (4.8.42), (4.8.43), and (4.8.44)) and $\beta(\alpha) \in (\alpha, \alpha_{\sup})$ (given by Equation (4.8.50)) such that

$$\liminf_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \geq \lim_{\tilde{\varepsilon}\to 0} p(\alpha) \sup_{\eta\in M_\sigma(\Sigma_A)} I(\widehat{\alpha}(\alpha,\tilde{\varepsilon})) = p(\alpha) I(\beta(\alpha)) \tag{4.8.53}$$

by Inequalities (4.8.46) and (4.8.14) and Equations (4.8.51) and (4.8.52). ■

This result gave us a weighted conditional variational principle as the lower bound for our large deviation principle. We will now combine our bounds (see Theorems 4.7.6 and 4.8.6) to prove our large deviation principle for $\frac{S_n f}{n}$ (see Theorem 4.1.6).

4.9 Finishing The Proof of Theorem 4.1.6

First, we restate the setting and consolidate our results for the reader. Fix $\lambda \in (\frac{1}{2}, 1)$. Recall the map $T_\lambda : (0,1] \to (0,1]$ given by

$$T_\lambda(x) := \begin{cases} \frac{x-\lambda}{1-\lambda} & \text{if } x \in (\lambda, 1] \\ \frac{x-\lambda^n}{\lambda(1-\lambda)} & \text{if } x \in (\lambda^n, \lambda^{n-1}] \text{ for each } n \geq 2, \end{cases} \tag{4.9.1}$$

shift space (Σ_A, σ) (see Equation (4.1.2)), and coding map $\pi : \Sigma_A \to (0,1]$. Recall that there exists a conjugacy (up to countably many points) such that

$$T_\lambda \circ \pi = \pi \circ \sigma.$$

Consider $\bar{N} := (N,N,N,...)$ for each $N \in \mathbb{N}$. Take the locally Hölder potentials $\phi_\lambda := -\log|T_\lambda \circ \pi|$ and $f : \Sigma_A \to \mathbb{R}$ such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty, \infty)$. We took

$$L := \lim_{N\to\infty} f(\bar{N}) \text{ and } \alpha_{\sup} := \sup_{\nu\in M_\sigma(\Sigma_A)} \left\{ \int f \, \mathrm{d}\nu \right\}.$$

Fix an $\alpha \in (L, \alpha_{\sup})$. For each $n \in \mathbb{N}$, let

$$X_\alpha^n := \left\{ x = (x_1, ..., x_{n-1}, x_n, x_{n+1}, ...) \in \Sigma_A : \frac{\sum_{i=0}^{n-1} f(\sigma^i(x))}{n} \geq \alpha \right\}.$$

A conditional variational principle will form part of our rate function (see Theorem 4.1.6). Hence, we define the function I as follows:

$$I(\gamma) := \sup_{\eta\in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, \mathrm{d}\eta + h(\eta) : \int f \, \mathrm{d}\eta \geq \gamma \right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$. To form our weight function, we define the following values and function. Recall the value

$$p_{\inf} := \frac{\alpha - L}{\alpha_{\sup} - L}.$$

Define the function β as

$$\beta(p,\alpha) := \frac{\alpha-(1-p)L}{p} \tag{4.9.2}$$

for each $p \in (p_{\inf}, 1]$. Consider the values $p(\alpha) \in (p_{\inf}, 1]$ and $\beta(p(\alpha),\alpha) := \beta(\alpha) \in (\alpha, \alpha_{\sup})$ such that

$$\beta(\alpha) = \frac{\alpha-(1-p(\alpha))L}{p(\alpha)} \tag{4.9.3}$$

and

$$\max_{p_{\inf}\le p\le 1} pI(\beta(p,\alpha)) = p(\alpha)I(\beta(\alpha)). \tag{4.9.4}$$

We restate Theorem 4.1.6 for the reader.

Theorem 4.9.1. *Fix $\lambda \in \left(\frac{1}{2},1\right)$. Recall the map T_λ given by Equation* (4.1.1)*, the shift space (Σ_A,σ), and the coding map $\pi : \Sigma_A \to (0,1]$. Let $\phi_\lambda := -\log|T_\lambda^{'} \circ \pi|$. Take $\bar{N} := (N,N,N,\ldots) \in \Sigma_A$ for each $N \in \mathbb{N}$. Assume that $f : \Sigma_A \to \mathbb{R}$ is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty,\infty)$. Recall that*

$$L := \lim_{N\to\infty} f(\bar{N}) \text{ and } \alpha_{\sup} := \sup_{\nu\in M_\sigma(\Sigma_A)} \left\{ \int f \,\mathrm{d}\nu \right\}.$$

Fix $\alpha \in \left(L,\alpha_{\sup}\right)$. Also, recall the expression:

$$p_{\inf} := \frac{\alpha - L}{\alpha_{\sup} - L}.$$

Then, there exists a function R, defined by $p(\alpha) \in (p_{\inf},1]$ and $\beta(\alpha) \in (\alpha,\alpha_{\sup})$ (see Equations (4.9.2), (4.9.3)*, and* (4.9.4)*), such that*

$$\lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) = R(\alpha) = p(\alpha)\left(\sup_{\eta\in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \,\mathrm{d}\eta + h(\eta) : \int f \,\mathrm{d}\eta \ge \beta(\alpha) \right\} \right) < 0.$$

Furthermore,

$$\lim_{\alpha\to L^+} R(\alpha) = 0.$$

Proof. Fix $\alpha \in \left(L,\alpha_{\sup}\right)$. By the upper and lower bounds of our large deviation principle (see Theorems 4.7.6 and 4.8.6), there exist $\beta(\alpha) \in (\alpha,\alpha_{\sup})$ and $p(\alpha) \in (p_{\inf},1]$ (given by Equations (4.9.3) and (4.9.4)) such that

$$p(\alpha)I(\beta(\alpha)) \le \lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \le p(\alpha)I(\beta(\alpha)).$$

Therefore, there exists a function R, defined by $\beta(\alpha) \in (\alpha,\alpha_{\sup})$ and $p(\alpha) \in (p_{\inf},1]$ (see Equations (4.9.2), (4.9.3), and (4.9.4)), such that

$$R(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) = p(\alpha)\left(\sup_{\eta\in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \,\mathrm{d}\eta + h(\eta) : \int f \,\mathrm{d}\eta \ge \beta(\alpha) \right\} \right) \tag{4.9.5}$$

for our fixed α.

We find that

$$\lim_{n\to\infty}\frac{1}{n}\log m(X_\alpha^n) = p(\alpha)I(\beta(\alpha)) \le p(\alpha)\mathscr{P}(\phi_\lambda) = p(\alpha)\log[4\lambda(1-\lambda)] < 0$$

because of Equation (4.9.5), $\mathscr{P}(\phi_\lambda) = \log[4\lambda(1-\lambda)]$ (see Theorem 4.2.1), and the variational principle. Furthermore,

$$\lim_{\alpha\to L^+} p(\alpha) = \lim_{\alpha\to L^+}\frac{\alpha - L}{\beta(\alpha)-L} = 0 \tag{4.9.6}$$

by Equation (4.8.49).

Hence,

$$\lim_{\alpha\to L^+} R(\alpha) = 0$$

by Equation (4.9.6). ■

We note that $p(\alpha)$ does not always equal to 1.

Lemma 4.9.2. *There exist $\alpha \in (L, \alpha_{\sup})$ such that $p(\alpha) < 1$.*

Proof. We will use a proof by contradiction and the variational principle to prove this lemma. By Theorem 4.1.6,

$$\lim_{\alpha\to L^+} R(\alpha) = 0. \tag{4.9.7}$$

Theorem 4.2.1 states that $\mathscr{P}(\phi_\lambda) < 0$.

Recall that the function I is defined as

$$I(\gamma) := \sup_{\eta\in M_\sigma(\Sigma_A)}\left\{\int \phi_\lambda\, \mathrm{d}\eta + h(\eta) : \int f\, \mathrm{d}\eta \ge \gamma\right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$. Take an $\alpha \in (L, \alpha_{\sup})$ such that the values $\beta(\alpha) \in (\alpha, \alpha_{\sup})$ and $p(\alpha) \in (p_{\inf}, 1]$ (see Equations (4.9.2), (4.9.3), and (4.9.4)) satisfy

$$\mathscr{P}(\phi_\lambda) < R(\alpha) = p(\alpha)I(\beta(\alpha)). \tag{4.9.8}$$

If $p(\alpha) = 1$, then

$$\mathscr{P}(\phi_\lambda) < I(\beta(\alpha)) = \sup_{\eta\in M_\sigma(\Sigma_A)}\left\{\int \phi_\lambda\, \mathrm{d}\eta + h(\eta) : \int f\, \mathrm{d}\eta \ge \beta\right\}. \tag{4.9.9}$$

Inequality (4.9.9) contradicts the variational principle for pressure (see Definition 2.3.23). Thus, $p(\alpha) < 1$ for this α. Therefore, there exist $\alpha \in (L, \alpha_{\sup})$ such that $p(\alpha) < 1$. ■

In the final section, we will discuss results, such as a large deviation principle on the interval, related to our large deviation principle (see Theorem 4.1.6).

4.10 Remark on Related Results

In the last section of my thesis, we analyse results that complement our large deviation principle, Theorem 4.1.6. Now, we form a large deviation principle on the interval. Fix $\lambda \in (\frac{1}{2}, 1)$. The Markov partition for T_λ is given by $\{(\lambda^n, \lambda^{n-1}]\}_{n\in\mathbb{N}}$. For each

$$y \in (0,1] = \bigcup_{n=1}^{\infty} (\lambda^n, \lambda^{n-1}],$$

there exists an element $x = \pi^{-1}(y)$ in Σ_A. Take $f : (0,1] \to \mathbb{R}$ such that $g = f \circ \pi$ is locally Hölder and $\lim_{x\to 0^+} f(x) \in (-\infty, \infty)$.

Our aim will be to explain how one could form a large deviation principle for $\frac{S_n f}{n}$. Denote

$$L := \lim_{x\to 0^+} f(x).$$

Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T_\lambda^i(y)) = L$$

for Lebesgue typical $y \in (0,1]$ by Theorem 4.1.4. There exists a value

$$\alpha_{int} := \sup_{\nu \in M_{T_\lambda}((0,1])} \left\{ \int f \, \mathrm{d}\nu \right\}.$$

Fix $\alpha \in (L, \alpha_{int})$. Then, take the set

$$X_\alpha^n := \left\{ y \in (0,1] : \frac{1}{n} \sum_{i=0}^{n-1} f(T_\lambda^i(y)) \ge \alpha \right\}$$

for each $n \in \mathbb{N}$.

The following function is a key component of our variational principle. Define the function I as

$$I(\gamma) := \sup_{\eta \in M_{T_\lambda}((0,1])} \left\{ \int \phi_\lambda \, \mathrm{d}\eta + h(\eta) : \int f \, \mathrm{d}\eta \ge \gamma \right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$.

Now, we will define the weight function $p(\alpha)$. Take the value

$$p_{\inf} := \frac{\alpha - L}{\alpha_{int} - L}.$$

Define the function β as follows:

$$\beta(p, \alpha) = \frac{\alpha - (1-p)L}{p} \tag{4.10.1}$$

for each $p \in (p_{\inf}, 1]$. Consider the values $p(\alpha) \in (p_{\inf}, 1]$ and $\beta(p(\alpha), \alpha) := \beta(\alpha) \in (\alpha, \alpha_{int})$ such that

$$\beta(\alpha) = \frac{\alpha - (1-p(\alpha))L}{p(\alpha)} \tag{4.10.2}$$

and

$$\max_{p_{\inf} \le p \le 1} pI(\beta(p,\alpha)) = p(\alpha)I(\beta(\alpha)). \tag{4.10.3}$$

Hence, there exists a function R, defined by $0 < p(\alpha) < 1$ and $\beta(\alpha) \in (\alpha, \alpha_{int})$, such that

$$\lim_{n\to\infty} \frac{1}{n} \log l(X_\alpha^n) = p(\alpha)\left(\sup_{\eta \in M_{T_\lambda}((0,1])} \left\{ \int \phi_\lambda \, d\eta + h(\eta) : \int f \, d\eta \ge \beta(\alpha) \right\} \right) = R(\alpha) < 0$$

for each $\alpha \in (L, \alpha_{int})$. If we had assumed that $L = -\infty$, then our large deviation principle would state: there exists a function R such that

$$\lim_{n\to\infty} \frac{1}{n} \log l(X_\alpha^n) = \sup_{\eta \in M_{T_\lambda}((0,1])} \left\{ \int \phi_\lambda \, d\eta + h(\eta) : \int f \, d\eta \ge \alpha \right\} = R(\alpha) < 0$$

for each $\alpha \in (-\infty, \alpha_{int})$. The proof for that principle would be similar to Inequality (4.1.15).

We could have also chosen a locally Hölder function f such that the limit L does not exist. Denote

$$\alpha_{\liminf} := \inf_{x\in\Sigma_A} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i(x)) \text{ and } \alpha_{\limsup} := \sup_{x\in\Sigma_A} \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i(x)).$$

We find the following large deviation estimates, which have similar proofs compared to Theorems 4.7.6 and 4.8.6, for each $\alpha \in (\alpha_{\liminf}, \alpha_{\limsup})$.

Define the function I as

$$I(\gamma) := \sup_{\eta \in M_{T_\lambda}((0,1])} \left\{ \int \phi_\lambda \, d\eta + h(\eta) : \int f \, d\eta \ge \gamma \right\}$$

for each $\gamma \in (\alpha, \alpha_{\sup})$.

Take the value

$$p_{\inf} := \frac{\alpha - L}{\alpha_{\sup} - L}.$$

There exist values $p(\alpha) \in (p_{\inf}, 1]$ and $\alpha < \widehat{\alpha}(\alpha) < \beta(\alpha) < \alpha_{\limsup}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) < p(\alpha)\left(\sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, d\eta + h(\eta) : \int f \, d\eta \ge \beta(\alpha) \right\} \right) \text{ and} \tag{4.10.4}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \ge p(\alpha)\left(\sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, d\eta + h(\eta) : \int f \, d\eta \ge \widehat{\alpha}(\alpha) \right\} \right). \tag{4.10.5}$$

Given our dynamical system (Σ_A, σ), a large deviation principle can be formed for level sets other than X_α^n. Consider the sequence $\bar{N} := (N, N, N, \ldots)$. Assume that f is a locally Hölder potential such that $\lim_{N\to\infty} f(\bar{N}) \in (-\infty, \infty)$. Denote

$$L := \lim_{N\to\infty} f(\bar{N}).$$

Fix any $\alpha \in (\alpha_{\inf}, L)$. We would take the set

$$Y_\alpha^n := \left\{ x = (x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots) \in \Sigma_A : \frac{\sum_{i=0}^{n-1} f(\sigma^i(x))}{n} \leq \alpha \right\}$$

for each $n \in \mathbb{N}$.

Define the function I as

$$I(\gamma) := \sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, \mathrm{d}\eta + h(\eta) : \int f \, \mathrm{d}\eta \leq \gamma \right\}$$

for each $\gamma \in (\alpha_{\inf}, \alpha)$. Define the function β as follows:

$$\beta(p, \alpha) = \frac{\alpha - pL}{1 - p} \tag{4.10.6}$$

for each $p \in (0, 1]$. Take the value

$$p_{\sup} := \frac{\alpha_{\inf} - \beta}{L - \beta}.$$

Consider the values $p(\alpha) \in (0, p_{\sup}]$ and $\beta(p(\alpha), \alpha) := \beta(\alpha) \in (\alpha_{\inf}, \alpha)$ such that

$$\beta(\alpha) = \frac{\alpha - p(\alpha)L}{1 - p(\alpha)} \tag{4.10.7}$$

and

$$\max_{0 < p \leq p_{\sup}} pI(\beta(p, \alpha)) = p(\alpha)I(\beta(\alpha)). \tag{4.10.8}$$

The proof of the following large deviation principle would be nearly identical to the one for Theorem 4.1.6. There exists a function R, defined by $p(\alpha) \in (0, p_{\sup}]$ and $\beta(\alpha) \in (\alpha_{\inf}, \alpha)$ (see Equations (4.10.6), (4.10.7), and (4.10.8)), such that

$$\lim_{n \to \infty} \frac{1}{n} \log m(Y_\alpha^n) = R(\alpha) = p(\alpha) \left(\sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, \mathrm{d}\eta + h(\eta) : \int f \, \mathrm{d}\eta \leq \beta(\alpha) \right\} \right) < 0$$

for each $\alpha \in (\alpha_{\inf}, L)$.

If we instead assume that $L = \infty$, we would use that $L > K$ for an arbitrarily large $K \in \mathbb{N}$ rather than L to form subsets for Y_α^n. Then, we would find the standard conditional variational principle

$$\lim_{n \to \infty} \frac{1}{n} \log m(Y_\alpha^n) = \sup_{\eta \in M_\sigma(\Sigma_A)} \left\{ \int \phi_\lambda \, \mathrm{d}\eta + h(\eta) : \int f \, \mathrm{d}\eta \leq \alpha \right\} < 0$$

for each $\alpha \in (\alpha_{\inf}, \infty)$. As done for X_α^n, similar large deviation principles for Y_α^n can be found on the interval and when L does not exist.

We will discuss the differences and similarities between finding large deviation estimates for $\frac{S_n f}{n}$ when $\lambda \in (\frac{1}{2}, 1)$ compared to when $\lambda \in (0, \frac{1}{2}]$. For each $\lambda \in \left(\frac{1}{2}, 1\right)$, the shift space (Σ_A, σ), associated to T_λ, fails to satisfy the BIP property. Hence, ϕ_λ does not have a Gibbs measure (see Sarig [Sar03]). For each $\lambda \in (0, 1)$, our reference measure m is not conservative (see Theorem A of

[BT12]), so we can neither form an inducing scheme nor use the Poincaré Recurrence Theorem. Finding large deviations results for $\frac{S_n f}{n}$ for a fixed $\lambda \in (0, \frac{1}{2}] \cup (\frac{1}{2}, 1)$ would involve using Egoroff's Theorem for the lower bound and pressure estimates for the upper bound.

Now, we compare the procedures for forming large deviation results when $\lambda \in (0, \frac{1}{2})$ and $\lambda = \frac{1}{2}$. Consider the dynamical systems $(T_\lambda, (0,1])$ and (Σ_A, σ) for any $\lambda \in (0, \frac{1}{2}]$. Let $f : \Sigma_A \to \mathbb{R}$ be a locally Hölder potential. We would aim to form a large deviation estimates for $\frac{S_n f}{n}$. Again, we would use $m = l \circ \pi$ as our reference measure and use that m is a conformal measure. Our upper bound for the measures of $n-$cylinders (see Proposition 4.2.2) will again be useful.

According to Page 176 of Bruin and Todd [BT12], m satisfies a form of the Poincaré Recurrence Theorem. Hence, we can create an induced space $\bar{\Sigma}_A$ from $\Omega_\lambda^{\complement}$. To do this, we would consider the set

$$\Delta_M := \bigcup_{i=1}^{M} [i]. \tag{4.10.9}$$

Take

$$\rho(\bar{\Sigma}_A) = \left\{x \in \Sigma_A : \exists \{n_i\}_{i \in \mathbb{N}} \text{ such that } \sigma^{n_i}(x) \in \Delta_M\right\}.$$

By a form of the Poincaré Recurrence Theorem, $m(\rho(\bar{\Sigma}_A)) = 1$. Hence. we would construct an induced space $\bar{\Sigma}_A$ from $\rho(\bar{\Sigma}_A)$.

Define the hitting time function as follows:

$$\tau(x) := \inf\left\{n \in \mathbb{N} : \sigma^n(x) \in \Delta_M\right\} + 1. \tag{4.10.10}$$

for each $x \in \Sigma_A$. We would take the alphabet

$$S := \bigcup_{n=1}^{\infty} \{[x_1, ..., x_{n-1}, x_n] : x_i \notin \Delta_M \text{ if } i \in [1, n-1] \text{ and } x_n \in \Delta_M\} \cup \Delta_M.$$

Then, we would consider the induced space

$$\bar{\Sigma}_A := \{b = ([b_1], [b_2], ...) : b_i \in S \text{ and } \rho(b) \in \Sigma_A\}$$

such that ρ is the natural projection. We would take the induced measure $\bar{m} = m \circ \rho$ and form a large deviation estimate on that space. Bounding the $\bar{m}-$measure of cylinders in $\bar{\Sigma}_A$ would be key (which would be a result similar to Proposition 4.2.2) and in turn, the results about pressure that follow would help us find our large deviation estimate. We can then project this result down from $\bar{\Sigma}_A$ to form our large deviation results for $\frac{S_n f}{n}$ on Σ_A.

However, the procedures for forming large deviation estimates when $\lambda \in (0, \frac{1}{2})$ and $\lambda = \frac{1}{2}$ have their differences. When $\lambda \in (0, \frac{1}{2})$, there exists an invariant ergodic measure $\mu \sim m$ (see the statement and proof of Theorem 1 from [BT12]). Then,

$$\lim_{n \to \infty} \frac{1}{n} S_n f(x) = \int f \, d\mu \tag{4.10.11}$$

for each $\mu-$typical $x \in \Sigma_A$. Consider the value $\alpha_{\sup}$ (see Equation (4.1.8)). Hence, we fix an $\alpha \in \left(\int f \, \mathrm{d}\mu, \alpha_{\sup}\right)$. We would find that there exists a function R such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \le R(\alpha) = \sup_{\nu \in M_\sigma(\Sigma_A)} \left\{ \int \tau \, \mathrm{d}\nu \left[\int \phi_\lambda \, \mathrm{d}\nu + h(\nu) \right] : \int f \, \mathrm{d}\nu \ge \alpha \text{ and } \nu(\Delta_M) > 0 \right\} < 0$$

for each $\alpha \in \left(\int f \, \mathrm{d}\mu, \alpha_{\sup}\right)$.

Now, we consider the case when $\lambda = \frac{1}{2}$. Assume that there exists a value L such that

$$L > \lim_{n\to\infty} \frac{1}{n} S_n f(x)$$

for all $m-$typical $x \in \Sigma_A$. We would find that there exists a function R such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(X_\alpha^n) \le R(\alpha) = \sup_{\nu \in M_\sigma(\Sigma_A)} \left\{ \int \tau \, \mathrm{d}\nu \left[\int \phi_\lambda \, \mathrm{d}\nu + h(\nu) \right] : \int f \, \mathrm{d}\nu \ge \alpha \text{ and } \nu(\Delta_M) > 0 \right\} < 0$$

for each $\alpha \in (L, \alpha_{\sup})$.

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